LUDWIG-MAXIMILIANS-UNIVERSTITÄT MÜNCHEN
Mathematische Fakultät

Lecture Notes for Functional Analysis 1

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CHAPTER 1

Topological and Metric Spaces

1. Topological Spaces - Basics

1.1. Definition. Let X be a set. Let $\mathcal{T} \in \mathcal{P}(X)$: $\mathcal{T}$ is a topology $:\Leftrightarrow$

- $\emptyset, X \in \mathcal{T}$
- $\mathcal{T}$ is closed under arbitrary unions.
- $\mathcal{T}$ is closed under finite intersections.

$(X, \mathcal{T})$ is called a topological space. A topological space is called Hausdorff if and only if $\forall x, y \in X,\ x \neq y\ \exists A, B \in \mathcal{T}\colon x \in A,\ y \in B,\ A \cap B = \emptyset$. If $\mathcal{T}_1,\ \mathcal{T}_2$ are topologies on X with $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\mathcal{T}_2$ is called finer than $\mathcal{T}_1$ and $\mathcal{T}_1$ coarser than $\mathcal{T}_2$

1.2. Example.

- indiscrete topology: $\mathcal{T} \coloneqq \{\emptyset, X\}$ coarsest top. on X.
- discrete topology: $\mathcal{T} \coloneqq \mathcal{P}(X)$ finest topology on X.
- standard topology (Euclidean) on $\mathbb{R}^n (n \in \mathbb{N})$: $A \subseteq \mathbb{R}^n$ open $:\Leftrightarrow$ $\forall x \in A,\ \exists \epsilon > 0,\ B_\epsilon(x) \subseteq A;\ B_\epsilon(x) \coloneqq \left\{ y \in \mathbb{R}^n : \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{\frac{1}{2}} < \epsilon \right\}$.

1.3. Definition. X a topological space and $A \subseteq X$ (not necessarily in $\mathcal{T}$). The relative topology on A is $\{A \cap C : C \in \mathcal{T}\}$

1.4. Remark. If A is not open in X, then $C \cap A$ need not be open in X

Example. Let $X \coloneqq \mathbb{R}$ with the standard topology, $A = [-1, 1]$ and $C =]-2, 0[\in \mathcal{T}$. Then $A \cap C = [-1, 0[$ is open in the relative topology, but not in the standard one.

1.5. Definition. X a topological space, $A \subseteq X$, $x \in X$

- A closed $:\Leftrightarrow A^c$ is open.
- $U \subset X$ is a neighbourhood of x $:\Leftrightarrow U$ is open and $x \in U$.
- $x \in X$ is an interior point of A $:\Leftrightarrow \exists$ neighbourhood U of x with $U \subseteq A$.
- X is a limit (accumulation) point of A $:\Leftrightarrow \forall$ neighbourhoods U of x: $U \cap A \neq \emptyset$.
- x is a boundary point $:\Leftrightarrow$ for all neighbourhoods U of x: $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$.
- $\partial A \coloneqq \{x \in X : x \text{ is a boundary point of } A\}$ is called the boundary of A.
- Closure of A: $\overline{A} = \{x \in X : x \text{ limit point of } A\}$.
- A dense in X $:\Leftrightarrow \overline{A} = X$.

1.6. Example. $A \coloneqq]-1, 0[\cup]0, 1]$; $\partial A = \{-1, 0, 1\}$; $\overline{A} = [-1, 1]$

Lemma. *Let $A \subseteq X$. Then:*

i) *$\overline{A}$ is closed,*
ii) *A closed $\Leftrightarrow \overline{A} = A$.*

Proof. i) Let $x \in \overline{A}^c$, i.e. x is no limit point of A meaning there is a neighbourhood U for x, such that $U \cap A = \emptyset$. If U is open then U is a neighbourhood of any $y \in U$ and $\forall y \in U : y$ is no limit point of A. So $U \subset A^c$ and therefore A^c open by Exercise 2a.

ii) $\overline{A} \supseteq A$ and Exercise 2b, which states A closed $\Leftrightarrow \overline{A} \subseteq A$. □

1.7. DEFINITION.
- Family $B \subset \mathcal{T}$ is a base for $\mathcal{T}$:⇔ $\mathcal{T} = \{$all unions of sets from $B\}$.
- $\mathcal{B}$ is a subbase for $\mathcal{T}$:⇔ {finite intersections of sets from B} is a base.
- Let $x \in X$. A family $\mathcal{N}$ of $\mathcal{T}$ is a neighbourhood base of x :⇔
  - i) Every $N \in \mathcal{N}$ is a neighbourhood of x.
  - ii) For all neighbourhoods U of x exists an $N \in \mathcal{N}$ with $N \subset U$.

1.8. REMARK. i) Let $\mathcal{C} \subset \mathcal{P}(X)$, then there exists a topology $\mathcal{T}_{\mathcal{C}}$ on X with $\mathcal{C}$ being a subbase for $\mathcal{T}_{\mathcal{C}}$ (including $\emptyset, X$, take finite intersections and arbitrary unions). It is the coarsest topology on X, such that every $S \in \mathcal{C}$. Jargon: $\mathcal{T}_{\mathcal{C}}$ is generated by $\mathcal{C}$.

ii) Example: $\mathbb{R}^n$ with the standard topology
- Fix $x \in \mathbb{R}^n$ then $\{B_{\frac{1}{k}}(x) : k \in \mathbb{N}\}$ is a neighbourhood base at x.
- $\{B_{\frac{1}{k}}(q) : k \in \mathbb{N}, q \in \mathbb{R}^n\}$ is a base.

1.9. DEFINITION. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. The topology on $X \times Y := \{(x, y) : x \in X, y \in Y\}$ generated by $\mathcal{C} := \{A \times B : A \in \mathcal{T}_X, B \in \mathcal{T}_Y\}$ is called the product topology.

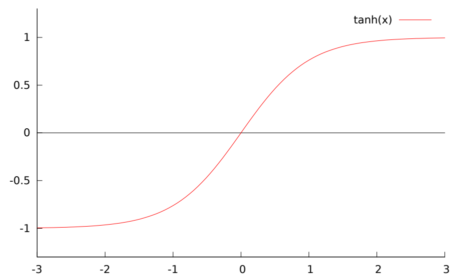
1.10. REMARK.
- Definition 1.9 works analogously for $\bigtimes_{\alpha \in I} X_\alpha$ with I an arbitrary index set.
- The standard topology on $\mathbb{R}^n$ is the product topology on $\mathbb{R}^n$ (cf. exercise).

2. Continuity and Convergence

1.11. DEFINITION. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces and $f: X \to Y$. For $B \subseteq Y$ let $f^{-1}(B) := \{x \in X : f(x) \in B\}$ be the inverse image or preimage
- f continuous :⇔ $B \in \mathcal{T}_Y$ implies $f^{-1}(B) \in \mathcal{T}_X$.
- f open :⇔ $f(A) \in \mathcal{T}_Y$ is open in Y for all $A \in \mathcal{T}_X$.
- f is a homeomorphism :⇔ f bijective, continuous and open (compatible with topologies).

1.12. EXAMPLE.

$X = \mathbb{R}$ with the standard topology, $Y =]-1, 1[$ with the relative topology, $f: X \to Y, x \mapsto \tanh(x)$ is a homeomorphism. (Details of the proof are worked out in Excercise 4 of Sheet 1)



1.13. DEFINITION. Let $(x_k)_{k \in \mathbb{N}} \subseteq X$ be a sequence in a topological space. $(x_k)_k$ converges to $x \in X$:⇔ for all neighbourhoods U of x exist at most finitely many $k \in \mathbb{N} : x_k \notin U$.

NOTATION. *We shall write* $\lim_{n\to\infty} x_n = x$ *or* $x_n \xrightarrow{n\to\infty} x$

1.14. REMARK.
- If X is Hausdorff, then limits are unique.
- The coarser a topology is, the easier it is to obtain convergence.

1.15. DEFINITION. Let X, Y be topological spaces

i) $f: X \to Y$ sequentially continuous :⇔ f conserves limits, i.e.

$$\lim_{n\to\infty} x_n = x \Rightarrow \lim_{n\to\infty} f(x_n) = f(x)$$

ii) X separable :⇔ X has a countable dense subset.
iii) X is 1st countable :⇔ $\forall x \in X$ exists a countable neighborhood base.
iv) X is 2nd :⇔ the countable topology has a countable (sub-)base.

REMARK. If a subbase is countable, then the corresponding base is countable.

PROOF. Countable unions of the countable sets

$$S_N := \{A = \text{intersection of } N \text{ set from the subbase}\}$$

are countable (Cantor's diagonal trick!) □

1.16. EXAMPLE.
- $\mathbb{R}^n$ (with std. topology) is separable, as $\mathbb{Q}^n$ is dense.
- $\mathbb{R}^n$ is 2nd (hence 1st) countable (Remark 1.8.1).

1.17. THEOREM. *X, Y topological spaces $f: X \to Y$*

i) *X 2^{nd} countable $\Rightarrow$ X 1^{st} countable and separable.*
ii) *f continuous $\Rightarrow$ f sequentially continuous.*
iii) *f is sequentially continuous and X is 1^{st} countable $\Rightarrow$ f is continuous.*

PROOF.

i) Let $\mathcal{B}$ be a countable base of the topology of X and let $x \in X$. Then $\mathcal{N}_x := \{B \in \mathcal{B} : x \in B\}$ is a neighbourhood base and X is first-countable. Separability: For $\varnothing \neq B \in \mathcal{B}$ pick $x_B \in \mathcal{B}$. Let $A := \{x_B \in X : B \in \mathcal{B}, B \neq \varnothing\}$. Claim: $\overline{A} = X$. Let $x \in X$ and U a neighbourhood of x (thus U open), then there exists $B \in \mathcal{B}: B \subset U$ thus $A \cap U \neq \varnothing$, i.e. x limit point of A.
ii) Let $x_k \xrightarrow{k\to\infty} x$, $V \subset Y$ a neighbourhood of $f(x)$ and $U := f^{-1}(V)$. f is continuous thus U is open and therefore a neighbourhood of x. Then $x_k \in U$ for all, except at most finitely many, $k \in \mathbb{N}$ and $f(x_k) \in V$ for almost all $k \in \mathbb{N}$, hence $f(x_k) \xrightarrow{k\to\infty} f(x)$.
iii) Assume f not continuous: $\exists V \subseteq Y$ open with $U := f^{-1}(V)$ not open, hence (c.f. Exercise 2a) there exists $x \in U$ such that for all neighbourhoods N of x: $N \setminus U \neq \varnothing$. Let $(N_k)_{k\in\mathbb{N}}$ be a neighbourhood base of x and set $\widetilde{N}_k := \bigcap_{j=1}^k N_j$. Hence $\widetilde{N}_k \setminus U \neq \varnothing$, $\forall k$. Now pick $x_k \in \widetilde{N}_k \setminus U$ and $\lim_{k\to\infty} x_k = x$ but $f(x_k) \notin V$, $\forall k$ and V is a neighbourhood of $f(x)$, i.e.: $\lim_{n\to\infty} f(x_n) \neq f(x)$. Contradiction. □

3. Metric Spaces

1.18. DEFINITION. i) Let $\varnothing \neq X$ be a set. A mapping $d: X \times X \to [0, \infty[$ is called a metric :⇔

a) $d(x,y) = 0 \Leftrightarrow x = y$ (Positivity)
b) $d(x,y) = d(y,x) \quad \forall x,y \in X$ (Symmetry)
c) $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x,y,z \in X$ (Triangle inequality)

ii) (X,d) is called a metric space.
iii) Let $Y \subseteq X$ then $d|_{Y\times Y}$ is called the restricted (induced) metric and $(Y, d|_{Y\times Y})$ is a metric space

1.19. EXAMPLE. (1) The standard metric on $\mathbb{R}^n$ or $\mathbb{C}^n$ $(n \in \mathbb{N})$:

$$d(x,y) := \left(\sum_{k=1}^n |x_k - y_k|^2\right)^{\frac{1}{2}} \qquad x = (x_1, \dots x_n),\ y = (y_1, \dots, y_n)$$

(2) The discrete metric on X: $d(x,y) := \begin{cases} 0, & x = y \\ 1, & \text{sonst} \end{cases}$

(3) If d is a metric on X, then $\tilde{d}(x,y) := \frac{d(x,y)}{1+d(x,y)}$ defines a metric on X

1.20. LEMMA. *If X is a metric space, then X is also a topological space with a 1^{st} countable, Hausdorff (with respect to the induced metric) topology generated by $\{B_{\frac{1}{n}}(x) : n \in \mathbb{N},\, x \in X\}$ where $B_{\frac{1}{n}} \coloneqq \left\{y \in X : d(x,y) < \frac{1}{n}\right\}$ is the open Ball of radius $\frac{1}{n}$ around x*

PROOF.
- $\{B_{\frac{1}{n}}(x) : n \in \mathbb{N}\}$ is a neighbourhood base of fixed $x \in X$ and countable.
- Let $x \neq y$, $\epsilon \coloneqq d(x,y) > 0$ thus $B_{\frac{\epsilon}{2}}(x) \cap B_{\frac{\epsilon}{2}}(y) \overset{\triangle\text{-ineq.}}{=} \varnothing$ □

1.21. REMARK. Lemma 1.20 allows to assign topological notions to a metric space like open sets, limit points, continuity, ...

1.22. COROLLARY. *X is a metric space $(x_n)_n \subset X$, $x \in X$, $A \subset X$*

(1) $\lim_{k\to\infty} x_k = x \Leftrightarrow \forall \epsilon > 0,\ \exists K : d(x_k, x) < \epsilon,\ \forall k \geq K \Leftrightarrow \lim d(x_k, x) = 0$
(2) $x \in \overline{A} \Leftrightarrow \{\exists (x_k)_k \subset A : \lim_{k\to\infty} x_k = x\}$
(3) *if Y is a metric space and $f\colon X \to Y$, then f is continuous* $\Leftrightarrow$

$$\lim_{k\to\infty} x_k = x \Rightarrow \lim_{k\to\infty} f(x_k) = f(x)$$

PROOF. Exercise □

1.23. WARNING. The "closed ball" $\overline{B}_\epsilon(x) \coloneqq \{y \in X : d(x,y) \leq \epsilon\} \neq \overline{B_\epsilon(x)}$ in general, e.g. let $\epsilon = 1$ in the discrete metric, then $\overline{B}_1(x) = X$, $\overline{B_1(x)} = \{x\}$

1.24. EXAMPLE. Space of continuous functions over $[0,1]$:

$$\mathcal{C}([0,1]) \coloneqq \{f\colon [0,1] \to \mathbb{C} : f \text{ continuous}\}$$

This space can be equipped with two different metrices $(f, g \in \mathcal{C}([0,1]))$

i) $\mathrm{d}_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$
- $\mathrm{d}_\infty(f,g) = 0 \Rightarrow f(x) = g(x),\ \forall x \in [0,1]$, i.e. $f = g$
- Symmetry is obvious
- Triangle inequality:

$$\mathrm{d}_\infty(f,g) = \sup_{x\in[0,1]} \underbrace{f(x) - g(x)}_{|f(x)-h(x)|+|h(x)-g(x)|} \leq \mathrm{d}_\infty(f,h) + \mathrm{d}_\infty(h,g)$$

ii) $\mathrm{d}_1(f,g) \coloneqq \int_0^1 |f(x) - g(x)|$
- Suppose $f \neq g \Rightarrow \exists x_0 \in\ ]0,1[\ : f(x_0) \neq g(x_0)$. Since f and g are continuous it follows, that there exist $\epsilon, \delta > 0$:

$$|f(x) - g(x)| < \epsilon,\ \forall x \in\ ]x_0 - \delta, x_0 + \delta[$$

giving us the positivity of d_1

$$\mathrm{d}_1(f,g) > \int_{x_0-\delta}^{x_0+\delta} \mathrm{d}_X\, |f(x) - g(x)| > 2\delta\epsilon > 0.$$

- Symmetry is clear.
- Triangle inequality is obvious.

1.25. DEFINITION. (X, d) metric space and $(x_k)_{k\in\mathbb{N}} \subset X$

i) $(x_k)_{k\in\mathbb{N}}$ is <u>Cauchy</u> $\Leftrightarrow \forall \epsilon > 0,\ \exists K \in \mathbb{N} :\ \mathrm{d}(x_n, x_l) < \epsilon,\ \forall k, l \geq K$.
ii) X is <u>complete</u> $:\Leftrightarrow$ every Cauchy sequence in X converges in X.

1.26. EXAMPLE.
- $\mathbb{R}^n$, $\mathbb{C}^n$ are complete with respect to the Euclidean metric.
- $\mathcal{C}([0,1])$ is complete with respect to d_∞, but is incomplete with respect to d_1.

1.27. REMARK.
- Convergent sequences are always Cauchy sequences.

- Let $(x_k)_{k\in\mathbb{N}}$ be Cauchy, then it is also a bounded sequence, i.e. there exists a bounded $A \subseteq X$: $x_k \in A$, $\forall k \in \mathbb{N}$.

1.28. DEFINITION. Let $\varnothing \neq A \subseteq X$.

- The diameter of A is $\underline{\text{diam}}(A) := \sup_{x,y\in A} \mathrm{d}(x,y) \in [0,\infty]$.
- A is $\underline{\text{bounded}}$ $:\Leftrightarrow \text{diam}(A) < \infty$.
- For $x \in X$: $\underline{\text{dist}}(x,A) := \inf_{y\in A} \mathrm{d}(x,y)$.

WARNING. Completeness is not a topological notion.

EXAMPLE. Let $X := (\mathbb{R},\mathrm{d})$ with the standard metric $\mathrm{d}(x,y) := |x-y|$ and $\hat{X} := (\mathbb{R},\hat{\mathrm{d}})$ with $\hat{\mathrm{d}}(x,y) = \tanh(x,y)$. The topologies induced by d and $\hat{\mathrm{d}}$ coincide, i.e. as topological spaces X and $\hat{X}$ are the same, but X is complete, $\hat{X}$ is incomplete.

$$\widetilde{B}_\epsilon(x) := \{y \in \mathbb{R} : \tanh(x-y) < \epsilon\} \;=\; \{y \in \mathbb{R} : |x-y| < \text{arctanh}(\epsilon)\} = B_{\text{arctanh}(\epsilon)}(x)$$

Also X is complete by construction of $\mathbb{R}$, but $\hat{X}$ is incomplete: $(x_n)_n = (n)_n$ is Cauchy in $\hat{X}$ ($\lim_{x\to\infty} \tanh(x) = 1$), but $\lim x_n \notin \mathbb{R}$.

1.29. LEMMA. *X is a complete metric space and $A \subseteq X \Rightarrow$*

$$A \textit{ complete } \Leftrightarrow A \textit{ closed}$$

PROOF. "$\Rightarrow$" Let $(x_{n_k}) \subset A$ be a Cauchy sequence. By the completeness of X: exists $x \in X : \lim_{n\to\infty} x_n = x$, so x is limit point of A. Since A closed it follows that $x \in A$. Hence A is complete.

"$\Leftarrow$" Let $x \in X$ be a limit point A, then by Corollary I.1.ii) exists $(x_n)_n \subset A$: $\lim_{n\to\infty} x_n = x$. $(x_n)_n$ is then Cauchy in A by Remark 27 which means $x \in A$ by the completeness of A. □

1.30. DEFINITION. Let (X,d_X), (Y,d_y) be metric spaces and $T\colon X \to Y$

- T is an $\underline{\text{isometry}}$ $:\Leftrightarrow \mathrm{d}_Y(T(x),T(y)) = \mathrm{d}_X(x,x')$, $\forall x,x' \in X$.
- (X,d_X) and (Y,d_Y) are $\underline{\text{isometric}}$ $\Leftrightarrow$ exists a bijective isometry $T\colon X \to Y$.

1.31. REMARK. Let $T\colon X \to Y$ be an isometry, then:

i) T is injective and continuous
ii) X and $T(X)$ are isometric

ad i): If $T(x) = T(x')$ then $\mathrm{d}_Y(T(x),T(x')) = d_X(x,x') = 0 \Rightarrow x = x'$

$$\lim_{n\to\infty} \mathrm{d}_X(x_n,x) = 0 \Rightarrow \lim_{n\to\infty} \mathrm{d}_X(T(x_n),T(x)) = 0 \Rightarrow \text{ i.e. } T \text{ sequentially continuous}$$

1.32. THEOREM. *Let X be a metric space, then there exists a complete metric space $\widetilde{X}$ and an isometry: $i\colon X \to \widetilde{X}$ with $i(X)$ dense in $\widetilde{X}$. The space $\widetilde{X}$ is unique up to homeomorphic copies.*

PROOF. a) Construct $(\widetilde{X},\tilde{\mathrm{d}})$.

b) Construct the isometry $i\colon X \to W \subset \widetilde{X}$, with W dense in $\widetilde{X}$.
c) Show that $\widetilde{X}$ is complete.
d) Proof the uniqueness of the construction.

ad a) Define equivalence relation on Cauchy sequences in X:

$$(x_n)_{n\in\mathbb{N}} = (x'_n)_{n\in\mathbb{N}} \Leftrightarrow \lim_{n\to\infty} \mathrm{d}(x_n,x'_n) = 0.$$

Set $\widetilde{X} := \{\text{eq. classes of Cauchy sequences in } X\}$. We shall write $(x_n)_n \in \tilde{x}$ meaning $(x_n)_{n\in\mathbb{N}}$ is a representative class of $\tilde{x}$.
Define the metric like this: $\tilde{\mathrm{d}}(x,y) = \lim_{n\to\infty} \mathrm{d}'(x_n,y_n)$ where $(x_n),(y_n) \subset \widetilde{X}$. Proof that $\tilde{d}$ is a well-defined [limits exist (i) and are independent of representatives (ii)] metric (iii):

i)
$$\begin{aligned}\tilde{\mathrm{d}}(x_n,y_n) &\le \tilde{\mathrm{d}}(x_n,y_m)+\tilde{\mathrm{d}}(y_m,y_n)\\ &\le \tilde{\mathrm{d}}(x_n,x_m)+\tilde{\mathrm{d}}(x_m,y_m)+\tilde{\mathrm{d}}(y_m,y_n)\end{aligned}$$
$$\Leftrightarrow \quad \tilde{\mathrm{d}}(x_n,y_n)-\tilde{\mathrm{d}}(x_m,y_m) \le \tilde{\mathrm{d}}(x_n,x_m)+\tilde{\mathrm{d}}(y_m,y_n)$$
$$\underset{m\leftrightarrow n}{\Leftrightarrow} \left|\underbrace{\tilde{\mathrm{d}}(x_n,y_n)}_{\alpha_n}-\underbrace{\tilde{\mathrm{d}}(x_m,y_m)}_{\alpha_m}\right| \le \tilde{\mathrm{d}}(x_n,x_m)+\tilde{\mathrm{d}}(y_m,y_n) \qquad (*)$$

$(x_n),(y_n)$ are Cauchy in X therefore (α_n) is Cauchy in $\mathbb{R}$ and thus by it's completeness has a limit.

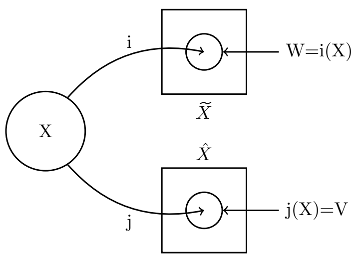
ii) Let $(x_n)\sim(x'_n)$ and $(y_n)\sim(y'_n)$ same reasoning as in (*) shows:
$$\left|\tilde{\mathrm{d}}(x_n,y_n)-\tilde{\mathrm{d}}(x'_n,y'_n)\right| \le \tilde{\mathrm{d}}(x_n,x'_n)+\tilde{\mathrm{d}}(y'_n,y_n) \xrightarrow{n\to\infty} 0.$$

iii) Symmetry and triangle inequality are clear. Suppose $\tilde{\mathrm{d}}(\tilde{x},\tilde{y})=0 \Rightarrow (x_n)\sim(y_n) \Leftrightarrow \tilde{x}=\tilde{y}$.

ad b) Define $i\colon X\to\widetilde{X}, b\mapsto\tilde{b}$, where $\tilde{b}$ is such that $(b,b,b,b,\dots)\in\tilde{b}$ and $W := i(X)$.
i is isometric: $\tilde{d}\,(i(b),i(b'))=\lim_{n\to\infty}\mathrm{d}(b,b')=\mathrm{d}(b,b')$.
Denseness: Let $\tilde{x}\in\tilde{X}$ and $\epsilon>0$. Pick $(x_n)_{n\in\mathbb{N}}$ Cauchy in X, then there exists N with $\mathrm{d}(x_n,x_m)<\epsilon$, $\forall n,m\ge N$. Define $\tilde{b}\in W$ by $(x_N,x_N,\dots)\in\tilde{b}$, then $\tilde{\mathrm{d}}(\tilde{b},\tilde{x})=\lim_{n\to\infty}\mathrm{d}(x_n,x_n)\le\epsilon$.

ad c) Completeness of $\widetilde{X}$: let $(\tilde{x}_n)_n\subset\widetilde{X}$ be Cauchy, for all $k\in\mathbb{N}$, exists $\tilde{z}_k\in W : \tilde{\mathrm{d}}(\tilde{x}_k,\tilde{z}_k)<\frac{1}{k}$ (W dense). Let $(z_k,z_k,\dots)\in\tilde{z}_k$ be the constant representative. Since i is an isometry:
$$\mathrm{d}(z_k,z_l)=\tilde{\mathrm{d}}(i(z_k),i(z_l))=\tilde{\mathrm{d}}(\tilde{z}_k,\tilde{z}_l)\le\tilde{\mathrm{d}}(\tilde{z}_k,\tilde{x}_k)+\tilde{\mathrm{d}}(\tilde{x}_k,\tilde{x}_l)+\tilde{\mathrm{d}}(\tilde{x}_l,\tilde{z}_l),$$
so $(z_1,z_1,\cdots)\equiv(z_n)_n$ is Cauchy in X. Now let $\epsilon>0$:
$$\tilde{\mathrm{d}}(\tilde{x},\tilde{x}_m)\le\tilde{\mathrm{d}}(\tilde{x},\tilde{z}_m)+\tilde{\mathrm{d}}(\tilde{z}_n,\tilde{x}_n)<\frac{1}{n}=\lim_{n\to\infty}\mathrm{d}(z_n,z_n)<\frac{1}{n}+\epsilon$$
Hence $\lim\tilde{\mathrm{d}}(\tilde{x}_n,x)=0$.

ad d) Suppose there were two completions $\widetilde{X}$ and $\hat{X}$ and two isometries $i\colon X\to W\subset\widetilde{X}$ and $j\colon X\to V\subset\hat{X}$, with W dense in $(\widetilde{X},\tilde{\mathrm{d}})$ and V dense in $(\hat{X},\hat{\mathrm{d}})$. Obviously $j\circ i^{-1}\colon W\to V$ is isometric and bijective. We need to extend $j\circ i^{-1}$ to an isometry T between $\widetilde{X}$ and $\hat{X}$ by employing denseness: for an arbitrary $\tilde{x}\in\widetilde{X}$ exist $(\tilde{z}_k)_{k\in\mathbb{N}}\subset W$ with $\lim_{k\to\infty}\tilde{z}_k=\tilde{x}$. So set $T(\tilde{x}):=\lim_{k\to\infty}(j\circ i^{-1})(\tilde{z}_k)=:\hat{z}_k=\hat{z}$



- The limit exists in $\widetilde{X}$, since $(\tilde{z}_k)$ is Cauchy in $\widetilde{X}$. j is an isometry, thus $(\hat{z}_k)$ is Cauchy in $\hat{X}$, which converges because $\widetilde{X}$ is complete.
- Independence of the approximating sequence: Let $\tilde{y}_k\xrightarrow{k\to\infty}\tilde{x}$, then:
$$\hat{d}\big((j\circ i^{-1})(\tilde{z}_k),(j\circ i^{-1})(\tilde{y}_k)\big)=\tilde{d}(\tilde{z}_k,\tilde{y}_k)\xrightarrow{k\to\infty}0$$
- $T\colon\colon\widetilde{X}\to\hat{X}$ isometric
$$\hat{\mathrm{d}}(T(\tilde{x}),T(\tilde{x}'))=\lim_{k\to\infty}\hat{\mathrm{d}}\,(T(\tilde{z}_k),T(\tilde{z}'_k))=\lim_{k\to\infty}\tilde{\mathrm{d}}(\tilde{z}_k,\tilde{z}'_k)=\mathrm{d}(\tilde{x},\tilde{x}')$$
- Bijectivity is clear. □

4. Example Sequence Spaces

1.33. DEFINITION (l^p-spaces). Let $p \in]0,\infty]$,

$$\underline{l^p} := \{x \in (x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{C}, \forall n \in \mathbb{N} \text{ and } \|x\|_p < \infty\}$$

where $\|x\|_p := \begin{cases} \left(\sum_n |x|_n^p\right)^{\frac{1}{p}}, & p \in]0,\infty[\\ \sup_{n\in\mathbb{N}} |x|_n, & p = \infty \end{cases}$ is the <u>p-norm</u> of x.

1.34. LEMMA. $\forall p \in [1,\infty]$ $d: l^p \times l^p \to [0,\infty[\,,\ (x,y) \mapsto \|x-y\|_p$ *is a metric.*

PROOF. $\mathrm{d}(x,y) = 0 \Leftrightarrow x = y$, symmetry obvious, $\triangle$-inequality see below. □

1.35. LEMMA. *Let* $p,q \in [1,\infty]$, $\frac{1}{p} + \frac{1}{q} = 1$ (*convention* $\frac{1}{\infty} = 0$) *then:*

i) *The <u>"dual pairing"</u>* $\langle x,y\rangle := \sum_{n=1}^{\infty} x_n y_n \in \mathbb{C}$ *is well defined* $\forall x \in l^p$, $y \in l^q$ *because of*

$$|\langle x,y\rangle| \le \sum_{n=1}^{\infty} |x_n|\,|y_n| \le \|x\|_p \|y\|_q \qquad \text{(\underline{Hölder})}$$

ii) *<u>Minkowski inequality:</u>* $\|x+y\|_p \le \|x\|_p + \|y\|_p$ *which implies*

$$\|x-y\|_p \le \|x-z\|_p + \|z-y\|_p \qquad \forall x,y,z \in l^p$$

PROOF. From the corresponding inequality in $\mathbb{C}^n$ (see e.g. Forster Analysis I):

$$\left|\sum_{k=1}^{n} x_k y_k\right| \le \sum_{k=1}^{n} |x_k|\,|y_k| \le \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}}, \ \forall n \in \mathbb{N} \text{ and } \lim n \to \infty$$ □

1.36. REMARK. Minkowski's inequality does not hold for $p < 1$, meaning d_p is not a metric in this case! We will therefore only consider l^p for $p \ge 1$.

1.37. THEOREM. i) l^P *is separable* $\forall p \in [1,\infty[$

ii) l^∞ *is not separable*

PROOF. i) $\forall n \in \mathbb{N}$ define

$M_n := \{x \in l^p : x_j \in \mathbb{Q} + i\mathbb{Q},\ \forall j \in \{1,\cdots,n\} \text{ and } x_j = 0,\ \forall j \ge n+1\}$

$M := \bigcup_{n\in\mathbb{N}} M_n$ is then countable.

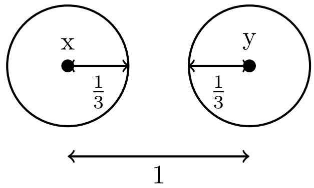
We will show $\overline{M} = l^p$, let $y \in l^p$, $\epsilon > 0$ $\exists n \in \mathbb{N} : \sum_{j=n+1}^{\infty} |y_j|^p < \frac{\epsilon^p}{2}$. Moreover $\exists x_j \in \mathbb{Q} + i\mathbb{Q}$ $(\forall j = 1,\cdots,n)$: $\sum_{j=1}^{n} |y_j - x_j|^p < \frac{\epsilon^p}{2}$ ($\mathbb{Q}$ dense in $\mathbb{R}$).
Let $x := (x_1,\cdots,x_n,0,\cdots) \in M$ then

$$\mathrm{d}_p(x,y) = \underbrace{\|x-y\|_p}_{<\epsilon} = \left(\underbrace{\sum_{j=1}^{n} |x_j - y_j|^p}_{\frac{\epsilon^p}{2}} + \underbrace{\sum_{j=n+1}^{\infty} |y_j|^p}_{\frac{\epsilon^p}{2}}\right)^{\frac{1}{p}} < \epsilon$$

ii) Let $M := \{x \in l^\infty : x_j \in \{0,1\},\ \forall j \in \mathbb{N}\}$. Claims:

1) M is uncountable.
2) $\mathrm{d}_\infty(x,y) = 1, \quad \forall x,y \in M$

ad 2) Clear by construction of M.
ad 1) $M \to [0,1]$, $x \mapsto \sum_{j=1}^{\infty} \frac{x_j}{2^j}$ is onto (binary represenation).



Idea: Suppose A were dense in l^∞, then $\forall x \in M$, there would exist $a_x \in A$: $\mathrm{d}_\infty(x, a_x) < \frac{1}{3}$ but $a_x \neq a_y$, $\forall x \neq y$. Therefore A must be uncountable. □

1.38. THEOREM. *l^p is complete $\forall p \in [1, \infty]$*

PROOF. Split the statement into two cases:

$p \in [1, \infty)$: Let $\left(x^{(n)}\right)_{n \in \mathbb{N}} \subset l^p$ be a Cauchy sequence. Let $\epsilon > 0$:

$$\exists N \in \mathbb{N},\ \forall n, m \geq N\colon \quad \sum_{i=J_1}^{J_2} \left|x_j^{(n)} - x_j^{(m)}\right|^p < \epsilon^p \quad \forall J_1 \leq J_2 \in \mathbb{N}. \tag{*}$$

Consequences:

(1) $J_1 = J_2 = j \Rightarrow \left|x_j^{(n)} - x_j^{(m)}\right|^p < \epsilon^p$, i.e. $\left(x_j^{(n)}\right)_{n \in \mathbb{N}}$ is Cauchy in $\mathbb{C}$ and by the completeness of $\mathbb{C}$ there exists $x_j \in \mathbb{C} : \lim_{n \to \infty} x_j^{(n)} = x_j,\ \forall j$

(2) $J_1 = 1$ and $m \to \infty$ in (*):

$$\sum_{j=1}^{J_2} \left|x_j^{(n)} - x_j\right|^p < \epsilon^p,\ \forall n \in \mathbb{N}_i,\ \forall J_2 \in \mathbb{N}$$

and hence there exists $x = (x_1, x_2, \cdots) \in l^p$ because

$$\|x\|_p = \left\|x - x^{(n)} + x^{(n)}\right\|_p \leq \left\|x - x^{(n)}\right\|_p + \left\|x^{(n)}\right\|_p < \infty$$

and we have $\lim_{n \to \infty} \left\|x - x^{(n)}\right\|_p = 0$.

$p = \infty$: Replace $\sum_{j=J_1}^{J_2}$ by $\sup_{i=J_1,\dots J_\infty}$ and the p's by 1 or ∞. □

5. Compactness

1.39. DEFINITION. Let X be a topological space $A \subseteq X$

- A compact $:\Leftrightarrow A \subseteq \bigcup_{\alpha \in I} B_\alpha$, where I is an arbitrary index set and B_α are open in X $\forall \alpha \in I$, then:

$$\exists N \in \mathbb{N} \text{ and } \alpha_1, \dots, \alpha_n \in I\colon A \subseteq \bigcup_{i=1}^{N} B_{\alpha_i}$$

- A sequentially compact $:\Leftrightarrow$ every sequence in A has a convergent subsequence.
- A relatively compact $:\Leftrightarrow \overline{A}$ is compact

REMARK. Some books (e.g. Bourbaki) use compactness only in conjunction with the Hausdorff property, otherwise they use the notion "quasi-compact".

1.40. THEOREM. *X_α topological space:*

i) *X 1^{st}-countable, A compactness $\Rightarrow$ A sequentially compact.*
ii) *X 2^{nd}-countable, A compactness $\Leftrightarrow$ A sequentially compactness.*

1.41. COROLLARY. *X is a separable metric space:*

$$\textit{Compactness} \ \Leftrightarrow\ \textit{Sequential compactness.}$$

REMARK. Corollary 41 holds for all metric spaces, i.e. separability is not needed, see e.g. Querenburg, Topologie, Satz 8.28.

PROOF COR 41). Use theorem 40 together with the fact that a separable metric space is 2^{nd}-countable. □

Proof Thm 40). i) Assume X is not sequentially compact, then there is a sequence $(x_n) \subset X$ without a converging subsequence.
Claim: $\forall x \in X$ exists a neighbourhood $U(x)$: $x_k \in U(x)$ for almost all k.
Reason: There exists a countable neighbourhood base $\{U_i\}_{i\in\mathbb{N}}$ of x. Let $V_m := \bigcap_{i=1}^m U_i$. Suppose the claim is false, then $\forall m \in \mathbb{N}$ exist infinitely many k with $x_k \in V_m$. Define a subsequence (x_{k_j}): $x_{k_m} \in V_m$ then $\lim_{j\to\infty} x_{k_j} = x$. This is a contradiction, thus by contradiction follows the Claim.
Now: Let $X = \bigcup_{x\in X} U(x)$ be an open cover, then by compactness follows $x = \bigcup_{j=1} U(a_j) \Rightarrow x_n \in X$ for only finitely many k ↯.

ii) We only need to prove "⇐" ("⇒" follows from i):
Assume there exists an open cover of X which does not have a finite subcover. X is 2^{nd}-countable, so every open cover is at most countable. Assume: $X = \bigcup_{j\in\mathbb{N}} c_j$ but without finite subcover. Pick $x_k \in X \setminus \left(\bigcup_{j=1}^n c_j\right) \forall n \in \mathbb{N}$. By assumption: $(x_n)_n$ has a convergent subsequence $(x_{n_k})_k$: $\lim_{k\to\infty} x_{n_k} = x$. Thus exists a $J \in \mathbb{N}$: $x \in C_J \Rightarrow x_{n_k} \in C_J$ for sufficiently large k, but $x_{n_k} \notin C_J$ for $n_k > J$ by construction ↯. □

Definition. A <u>metrisable</u> space is a topological space $(X, \mathcal{T})$ that is homeomorphic to a metric space (X, d), i.e. there is a metric d, such that $\mathrm{d}: X \times X \to [0, \infty)$ induces the topology $\mathcal{T}$. Here X is a compact, metric space.

1.42. Theorem. *X is a compact Hausdorff space:*
i) *X is metrisable $\Rightarrow$ X is separable.*
ii) *$A \subseteq X$ compact $\Leftrightarrow$ A closed.*

Proof. i) $\forall k \in \mathbb{N}$ $X = \bigcup_{x\in X} B_{\frac{1}{k}(x)}$. So $X = \bigcup_{n=1}^{N_k} B_{\frac{1}{k}}(x_k^{(n)})$ for some $N_k \in \mathbb{N}$ and $x_k^{(n)} \in X$ by compactness of X. Define

$$A := \left\{ x_k^{(n)} \in X : n \in \{1, \cdots N_k\};\ k \in \mathbb{N} \right\}.$$

A is countable. $\overline{A} = X$ is true, because $\forall x \in X: \forall \epsilon > 0,\ B_\epsilon(x) \cap A \neq \varnothing$.

ii) "⇐" Let $A \subseteq \bigcup_{i\in I} U_i$ be covered, by U_i open. A^c is open $\Rightarrow X = A^c \cup \left(\bigcup_{i\in I} U_i\right)$ is an open cover of X, by compactness $X = A^c \cup \left(\bigcup_{k=1}^N U_{i_k}\right)$. Therefore $A \subseteq \bigcup_{i_n=1}^N U_{i_k}$ is a finite subcover, i.e. A compact.

"⇒" Show A^c is open: $\forall y \in A$, $\forall xinA^c$ exists a neighbourhood U_y of y and a neighbourhood V_x of x: $U_y \cap V_x = \varnothing$.

$$A \subseteq \bigcup_{y\in A} U_y \Rightarrow A \subseteq \bigcup_{n=1}^N U_{y_n} \qquad \text{since } A \text{ is compact.}$$

Define $V := \bigcap_{n=1}^N V_{y_n}$ open and $V \cap A \neq \varnothing$.
Summary: for all $x \in A^c$ exists a neighbourhood V of x: $V \subseteq A^c$. Hence A^c is open, i.e. A closed! □

Warning. Let $A \subseteq X$, then A bounded and closed $\not\Rightarrow$ A compact, in general.

1.43. Example. $\overline{B_1}(0)$, the closed unit sphere in l^p $\forall p \in [1, \infty]$ is bounded and closed,

$$\overline{B_1}(0) := \{y \in l^p : \|y\|_p \leq 1\} \qquad \left(= \overline{B_1(0)}\right)$$

Closedness: Let x be a limit point of $\overline{B_1}(x)$: $\exists \left(y^{(n)}\right)_n \subset \overline{B}_1(0)$: $\left\|y^{(n)} - x\right\|_p \xrightarrow{n\to\infty} 0$. Let $\epsilon > 0$ then

$$\|x\|_p \leq \underbrace{\|x-y\|_p}_{<\epsilon} + \underbrace{\|y^n\|_p}_{\leq 1} \xrightarrow{\epsilon \text{ arb.}} \|x\|_p \geq 1 \Rightarrow x \in \overline{B}_1(0).$$

Moreover $\overline{B}_1(0)$ is not compact: consider the sequence $e_N \coloneqq (0,\cdots,0,1,0,\cdots,0) \in \overline{B}_1(0)$. Obviously $(e_n)_n \subset \overline{B}_1(0)$, but $\forall n \neq m$: $\|e_n - e_m\|_p = \begin{cases} 2^{\frac{1}{p}}, & p < \infty \\ 1, & p = \infty \end{cases}$

Thus by Lem. 20 and Thm. 40i) it follows that $\overline{B}_1(0)$ is compact.

1.44. Theorem (Tychonoff). *Let I be an index set, X_α a compact topological space $\forall \alpha \in I$. Then the Cartesian product*

$$\bigtimes_{\alpha\in I} X_\alpha \coloneqq \{\chi : I \to \bigcup_{\alpha\in I} X_\alpha : \chi(\alpha) \in X_\alpha\}$$

is compact in the product topology which is the coarsest topology on $\times_{i\in I} X_\alpha$ with the property, that $\forall \alpha \in I$ the projection $\pi_\alpha : \times_{\beta\in I} X_\beta \to X \setminus \alpha$, $\chi \mapsto \chi(\alpha)$ is continuous. In other words, it is generated by $\{\pi_\alpha^{-1}(B_\alpha) : B_\alpha \subseteq X_\alpha$ open and $\alpha \in I\}$

Proof. Jähnich et. al. □

1.45. Definition. X, Y topological space

$$\underline{\mathcal{C}_Y(X)} \coloneqq \{f : X \to Y : f \text{ continuous}\}, \quad \underline{\mathcal{C}(X)} \coloneqq \mathcal{C}_{\mathbb{C}}(X).$$

1.46. Theorem. *Let X, Y topological spaces, X compact, $f \in \mathcal{C}_Y(X)$ then*

i) *$f(X)$ compact.*
ii) *If also X, Y Hausdorff and if also f bijective, then f is a homeomorphism.*
iii) *If $Y = \mathbb{R}$, then f attains its maximum and minimum.*

Proof. i) Let $f(x) = \bigcup_{\alpha\in I} B_\alpha$, B_α open in $Y \Rightarrow X = \bigcup_{\alpha\in I} f^{-1}(B_\alpha)$. Since X is compact $X = \bigcup_{n=1}^N f^{-1}(B_\alpha) \Rightarrow f(X) \subseteq \bigcup_{n=1}^N B_{\alpha_n}$

ii) Exercise
iii) Exercise □

6. Example: Spaces of Continous Functions

Notation. *In this subsection:*

- *X is compact and Hausdorff. $Y = \mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$*
- *Equip $C_{\mathbb{K}}(X)$ with the* sup*-metric: $d_\infty(f,g) \coloneqq \sup_{x\in X} |f(x) - g(x)|$*

1.47. Theorem. *$C_{\mathbb{K}}(X)$ is complete*

Proof. Let $(f_n)_n \subset C_{\mathbb{K}}(X)$ be Cauchy, let $\epsilon > 0$ then exists an $N \in \mathbb{N}$:

$$\forall n, m \neq N : \sup_{x\in X} |f_n(x) - f_m(x)| < \epsilon \tag{*}$$

So $(f_n(x))_n \subset \mathbb{K}$ is Cauchy at every point $x \in X$ and by $\mathbb{K}$'s completeness the limit $f(x)$ of $f_n(x)$ exists for all x. Now:

$$\begin{aligned} \sup_{x\in X} |f(x) - f_N(x)| &= \sup_{x\in X} \lim_{n\to\infty} |f_n(x) - f_N(x)| \le \sup_{x\in X} \sup_{n\ge N} |f_n(x) - f_N(x)| \\ &\le \sup_{n\ge N} \sup_{x\in X} |f_n(x) - f_N(x)| \\ &\le \sup_{n\ge N} \mathrm{d}(f_n, f_N) < \epsilon \end{aligned} \tag{*}$$

Thus (f_n) converges to f in d_∞ uniformly. No we need to show that f is continuous, i.e. $f \in C_{\mathbb{K}}(X)$. We prove: for all $V \subseteq \mathbb{K}$ open: $f^{-1}(V)$ is open, by showing: for all $x \in f^{-1}(V)$, exists a neighbourhood $U(x)$ of $x : U_x \subset f^{-1}(V)$.
So let $V \subseteq \mathbb{K}$ open be given and fix an arbitrary $x \in f^{-1}(V)$. Now V is open, therefore there is an $\epsilon > 0 : B_{3\epsilon}(f(x)) \subset V$. Choose N so (*) holds and an arbitrary $y \in U_x \coloneqq f_N^{-1}(B_\epsilon(f_N(x))$:

$$|f(x) - f(y)| \le |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \le 3\epsilon$$

This means $f(U_x) \subseteq B_{3\epsilon}(f(x))$, i.e. $U_x \subseteq f^{-1}(V)$ and $x \in U_x$ □

1.48. Theorem. *X is metrisable $\Leftrightarrow$ $C_{\mathbb{K}}(X)$ separable*

Proof. See e.g. Bourbaki, Elements of Mathematics, General topology, part 2, sec. X.3.3 Thm. 1. We only prove "$\Rightarrow$": For $m, n \in \mathbb{N}$ let

$$G_{m,n} \coloneqq \left\{ f \in C_{\mathbb{K}}(X) : \quad \mathrm{d}(x, x') \le \frac{1}{m} \Rightarrow |f(x) - f(x')| \le \frac{1}{n} \quad \forall x, x' \in X \right\} \tag{*}$$

X is compact, therefore $f \in C_{\mathbb{K}}(X)$ is uniformly continuous, implying

$$\forall n \in \mathbb{N} : C_{\mathbb{K}}(X) = \bigcup_{m \in \mathbb{N}} G_{m,n}. \tag{**}$$

Also by compactness:

$$\forall m \in \mathbb{N}\ \exists a_1, \cdots, a_{p(m)} \in X : \quad X = \bigcup_{k=1}^{p(m)} B_{\frac{1}{m}}(a_k).$$

Let $\{\mathfrak{r}_\nu : \nu \in \mathbb{N}\} \subseteq \mathbb{K}$ be dense. Fix $m \in \mathbb{N}$ and $\phi \colon \{1, \cdots, p(m)\} \to \mathbb{N}$ (i.e.: $\phi \in \mathbb{N}^{p(m)}$). Let

$$G_{mn}^{(\phi)} \coloneqq \left\{ f \in G_{mn} : \left| f(a_n - \mathfrak{r}_{\phi(k)} \right| \le \frac{1}{n} \quad \forall k \le p(m) \right\} \tag{***}$$

$$\varPhi_{mn} \coloneqq \left\{ \phi \in \mathbb{N}^{p(m)} : G_{mn}^{(\phi)} \neq \varnothing \right\},$$

For each $\phi \in \varPhi_{mn}$, let g_ϕ be one element of $G_{mn}^{(\phi)}$. Now define $L_{mn} \coloneqq \{g_\phi : \phi \in \varPhi_{mn}\}$; this is countable. For all $f \in G_{mn}$ exists a $\phi_f \in \varPhi_{mn}$ with $f \in G_{mn}^{(\phi_f)}$ and for each $x \in X$ let $k_x \in \{1, \ldots, p(m)\} : x \in B_{\frac{1}{m}}(a_{k_x})$. Then for $x \in X$ arbitrary:

$$\begin{aligned} \left| f(x) - g_{\phi_f}(x) \right| \le |f(x) - f(a_{k_x})| + \left| f(a_{k_x}) - \mathfrak{r}_{\phi_f(k_x)} \right| + \left| \mathfrak{r}_{\phi_f(k_x)} - g_{\phi_f}(a_{k_x}) \right| \\ + \left| g_{\phi_f}(a_{k_x}) - g_{\phi_f}(x) \right| \end{aligned}$$

By definition of G_{mn} (**) and $G_{mn}^{(\phi)}$ (***)

$$\le \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \frac{1}{n} = \frac{4}{n}$$

Now $g_{\phi_f} \in L_{mn}$, so we have proven $\mathrm{d}_\infty(f, g_{\phi_f}) \le \frac{4}{n}$ $\forall f \in G_{mn}$ which means $\mathrm{dist}_\infty(f, L_{mn}) \le \frac{4}{n}$ $\forall f \in G_{mn}$. With (*) this means: $\bigcup_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} L_{mn} \subset C_{\mathbb{K}}(x)$ is a dense, countable subset and $C_{\mathbb{K}}$ is thus separable. □

This theorem can also be proven with the help of the famous approximation theorem of Stone-Weierstraß. In order to state it some definitions are in order.

1.49. Definition.
- i) A ($\mathbb{K}$-)vector space is a <u>($\mathbb{K}$-)algebra</u> $A :\Leftrightarrow A$ has a multiplication $A \times A \to A$, where the distributive laws hold (Example: $C_{\mathbb{K}}(X)$ is an algebra)
- ii) A subspace B of an algebra A is a <u>subalgebra</u> $:\Leftrightarrow B$ is closed under multiplication
- iii) A subset $B \subset C_{\mathbb{K}}(X)$ <u>separates points</u> $:\Leftrightarrow \forall x, y \in X$, $x \neq y$ $\exists A \in B$ with $f(x) \neq f(y)$

1.50. Theorem (Stone-Weierstrass). *Let $B \subset C_{\mathbb{K}}(X)$ be a subalgebra, $B \ni \mathbb{1} \coloneqq (x \mapsto 1)$, closed with respect to d_∞ and separating points. If $\mathbb{K} = \mathbb{C}$ assume in addition, that B is closed under complex conjugation. Then $B = C_{\mathbb{K}}(X)$*

Proof. See Reed, Simon Appendix to Section 3. □

1.51. Corollary (Weierstraß). *Let $d \in \mathbb{N}$, $U \subset \mathbb{R}^d$ be compact, then the set of all polynomials over U is dense in $C_{\mathbb{K}}(X)$ with respect to d_∞. In particular $C_{\mathbb{K}}(U)$ is separable.*

PROOF. (Given Thm. 54): Let B_0 be the $\mathbb{K}$-algebra generated by the monomials over U, $U \to \mathbb{K}$, $x \mapsto x_\alpha^n$ where $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $\alpha \in \{1, \cdots, d\}$, $n \in \mathbb{N}_0$ and let $\overline{B}_0$ be the closure with respect to d_∞. Then by Theorem 50 $\overline{B}_0 = C_{\mathbb{K}}(U)$.
Separability: Take the $\mathbb{Q}$-algebra (for $\mathbb{K} \coloneqq \mathbb{R}$), or $\mathbb{Q}^+ + i\mathbb{Q}$-Algebra for $\mathbb{K} = \mathbb{C}$, instead of the $\mathbb{K}$-algebra, then the polynomials with "rational coefficients" are dense. □

Completeness of $\mathcal{C}_{\mathbb{K}}(X)$ relied on the fact, that the uniform limit of continuous functions is continuous. If there is no uniformity of the limit in x, but uniformity of the continuity in n instead, then the limit will still be continuous...

1.52. DEFINITION. Let X be a metric space (not necessarily compact) and F a family of continuous functions $X \to \mathbb{K}$

$$\underline{F \text{ equicontinuous}} :\Leftrightarrow \forall \epsilon > 0\, \forall x \in X\, \exists \delta > 0\, \forall f \in F : f\left(B_\delta(x)\right) \subseteq B_\epsilon\left(f(x)\right)$$
$$\underline{F \text{ uniformly equicontinuous}} :\Leftrightarrow \forall \epsilon > 0\, \exists \delta > 0\, \forall x \in X\, \forall f \in F : f\left(B_\delta(x)\right) \subseteq B_\epsilon\left(f(x)\right)$$

1.53. REMARK.
i) X compact metric space, then F is equicontinuous if and only if F is uniformly equicontinuous.
ii) Examples for $X = [0,1]$:
   - $F = \{x \mapsto \cos(x/n), n \in \mathbb{N}\}$ is uniformly equicontinuous.
   - $F = \{x \mapsto x^{\frac{1}{n}}, n \in \mathbb{N}\}$ is not equicontinuous.

1.54. THEOREM. *Let X e a metric space and $(f_n)_{n\in\mathbb{N}}$ a sequence of functions $f_n\colon X \to \mathbb{K}$, which is equicontinuous (i.e. $\{f_n : n \in \mathbb{N}\}$ is equicontinuous). If the pointwise limit* $\lim_{n\to\infty} f_n(x) = f(x)$ *exists for all $x \in X$, then f is continuous.*

PROOF. Let $\epsilon > 0$ and $x \in X$, then by equicontinuity exists a $\delta > 0$:

$$|f_n\left(B_\delta(x)\right)| \subset B_\epsilon\left(f_n(x)\right)\ \forall n \in \mathbb{N} \ \Leftrightarrow\ \forall y \in B_\delta(x) \forall n \in \mathbb{N}\ |f_n(x) - f_n(y)| < \epsilon$$

Taking the limit and by pointwise convergence we get

$$\mathrm{d}(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Please take note, that δ is independent of m. □

The requirements for Theorem 54 will now be relaxed:

1.55. LEMMA. *Assume the same situation as in Theorem 54, but instead of* $\lim_{n\to\infty} f_n(x) = f(x)\ \forall x \in X$, *we only require that there exists $D \subseteq X$ dense and* $\lim_{n\to\infty} f_n(x) = f(x)\ \forall x \in D$. *Then the conclusion of Theorem 54 still holds.*

PROOF. Exercise
i) Define $f(x)\, \forall x \in X$.
ii) Show f is continuous. □

1.56. LEMMA. *Let X be a compact metric space and $(f_n)_n \subset \mathcal{C}_{\mathbb{K}}(X)$ an equicontinuous sequence. If* $\lim_{n\to\infty} f_n(x)$ *exists for all $x \in D$ where $D \subseteq X$ is dense, then there exists $f \in f(x) : \lim_{n\to\infty} d_\infty(f_n, f) = 0$*

PROOF. By Lemma 55 there exists $f \in \mathcal{C}_{\mathbb{K}}(X) : \lim_{n\to\infty} f_n(x) = f(x)\ \forall x \in X$. We need to show the uniformity of the convergence. Let $\epsilon > 0$; since X is compact $\{f_n\}$ is uniformly equicontinuous(by Remark 53) (a), f is uniformly continuous (b) and we can cover X by finitely many $B_\delta(a_l)$ (c). So $\exists N \in \mathbb{N} \forall x \in X$:

$$|f(x) - f_n(x)| \leq \underbrace{|f(x) - f(a_{l_x})|}_{\leq\epsilon \text{ a),b)}} + \underbrace{|f(a_{l_\epsilon} - f_n(a_{l_x})|}_{<\epsilon} + \underbrace{f_n(a_{l_x} - f_n(x)}_{\leq\epsilon} \leq 3\epsilon$$

The second inequality holds because of pointwise convergence and c) and the third because of a), b) and taking the limit $n \to \infty$. Overall we get

$$\forall \epsilon > 0\, \exists N \in \mathbb{N} : \lim_{n\to\infty} \mathrm{d}_\infty(f_n, f) = 0\ \forall x \in X$$ □

1.57. Theorem (Arzela-Ascoli). *Let X be a compact metric space and $(f_n) \subset \mathcal{C}_{\mathbb{K}}(X)$ a uniformly bounded, equicontinuous sequence* $(\sup_{n\in\mathbb{N}} \sup_{x\in X} |f_n(x)| < \infty)$ *then there exists a subsequence of f_n which is uniformly converging. Equivalently every uniformly bounded, equicontinuous subset $E \subset \mathcal{C}_{\mathbb{K}}(X)$ is relatively compact.*

Proof. Since $\mathcal{C}_{\mathbb{K}}(X)$ is a separable, metric space hence compactness and sequential compactness are interchangable by Corollary 41, so we only prove the 1st statement. By Theorem 41 i) X is separable. Let $\{a_l : l \in \mathbb{N}\}$ be a countable dense subset in X. By Uniform boundedness we have

$$\exists c < \infty \ \ (f_n(a_l)) < c \quad \forall n \in \mathbb{N}, \ \forall l \in \mathbb{N}.$$

By Bolzano-Weierstrass $(f_n(a_l))_{n\in\mathbb{N}}$ has a converging subsequence $\left(f_{n_j^{(l)}}(a_l)\right)$. It can be chosen, in a way that $\left(n_j^{(l_1-1)}\right)_i \le \left(n_j^{(l)}\right)_i$.
Diagonal sequence trick: Choose $n_i = n_i^{(i)}$ then $(n_j)_{j<l} \subset \left(n_j^{(l)}\right)_j$ is a subsequence, so that $\lim_{j\in\mathbb{N}} f_{n_j}(a_l)$ exists for all $l \in \mathbb{N}$ by Lemma 56. □

1.58. Remark. Both compactness of X and equicontinuity of (f_n) are essential in the above Theorem (see e.g. Remark to Theorem 2.4.4. in Dobrowolski)

7. Baire's Theorem

1.59. Remark. Let $A_1, A_2 \subset X$ be open and dense in a metric space X, then $A_1 \cap A_2$ is open and dense.

Proof. Exercise □

1.60. Theorem (Baire). *Let X be a complete metric space and $\forall n \in \mathbb{N}$ let $A_n \subseteq X$ be open and dense $\Rightarrow \bigcap_{n\in\mathbb{N}} A_n$ is dense.*

Proof. Let $D := \bigcap_{n\in\mathbb{N}} A_n$. Für $x_0 \in X$ arbitrary and $\epsilon > 0$. We show $D \cap \underline{B_{\epsilon_0}(x_0)} \neq \varnothing$. Clearly $A_1 \cap B_\epsilon(x_0)$ is open; therefore $\exists x_1 \in A_1$ and $\epsilon_1 \in [0, \frac{\epsilon}{2}[$ with $\overline{B_{\epsilon_1}(x_1)} \subseteq A_1 \cap B_\epsilon(x_0)$. Now iterate this $\Rightarrow \exists (\epsilon_n)_n \in \mathbb{R}^{\mathbb{N}}$ and $(x_n)_n \subset X$ sequences:

i) $\epsilon_{n+1} \in \left]0, \frac{\epsilon_n}{2}\right[\ \forall n \in \mathbb{N} \Rightarrow \epsilon_n < \frac{\epsilon_0}{2^n}$
ii) $\overline{B_{\epsilon_{n+1}}(x_{n+1})} \subseteq A_{n+1} \cap B_{\epsilon_n}(x_n) \subseteq A_{n+1} \cap A_n \cap \ldots \cap B_{\epsilon_0}(x_0)$

1) and 2) give: $\forall N \in \mathbb{N}\ \forall n \in \mathbb{N} : x_n \in B_{\epsilon_n}(x_N) \subseteq B_{\frac{\epsilon_0}{2^n}}(x)$ so $(x_n)_{n\in\mathbb{N}}$ is Cauchy and since X is complete, there exists an $x \in X : \lim_{n\to\infty} x_n = x$ with $x \in B_{\frac{\epsilon_0}{2^N}}(x_n)\ \forall N \in \mathbb{N}$
In all this give us $x \in D \cap B \subseteq (x_0)$ □

1.61. Remark. i) Completeness is essential for the theorem. Consider $X = \mathbb{Q}$ with the induced metric of $\mathbb{R}$ and $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$, $A_n := \mathbb{Q} \setminus \{q_n\}$ dense and open in $\mathbb{Q}$, but $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$.
ii) Openness is not claimed in Baire's theorem and is not true in general.

1.62. Definition. Let X be a topological space and $A \subseteq X$

i) A is a $\underline{G_\delta\text{-set}}$ $:\Leftrightarrow$ A is the countable intersection of open sets.
ii) A is nowhere dense $:\Leftrightarrow$ $\overline{A}$ has no interior points
iii) A is meagre (or of 1st category) $:\Leftrightarrow$ A is a countable union of nowhere dense sets.
iv) A is nonmeagre (or of 2nd category) $:\Leftrightarrow$ A is not meagre.

1.63. Example. $\mathbb{Q}$ is meagre in $\mathbb{R}$.

1.64. Lemma. *Let X be a topological space.*

i) *$A \subseteq X$ is nowhere dense $\Leftrightarrow$ $\overline{A}^c$ dense.*
ii) *A is meagre in X and $B \subset A \Rightarrow B$ meagre.*

iii) *$A_n \subseteq X$ is meagre $\forall n \in \mathbb{N} \Rightarrow \bigcup_{n\in\mathbb{N}} A_n$ is meagre.*
iv) *If A_n is nowhere dense $\forall n \in \mathbb{N} \Rightarrow \bigcup_{n\in\mathbb{N}} A_n$ has no interior points.*

1.65. Lemma. *Let X be a topological space. Then the following are equal.*
i) *$A_n \subseteq X$ open, dense $\Rightarrow \bigcap_{n\in\mathbb{N}} A_n$ dense.*
ii) *$A_n \subseteq X$ is a dense G_δ-set $\Rightarrow \bigcap_{n\in\mathbb{N}} A_n$ is a dense G_δ-set.*
iii) *$A \subseteq X$ open and $A \neq \varnothing \Rightarrow A$ is non-meagre.*
iv) *$A \subseteq X$ is meagre $\Rightarrow A^c$ is dense.*

Proof. Exercise 16 □

1.66. Corollary. *In a complete metric space i) - iv) hold. In particular, if $X \neq \varnothing \Rightarrow X$ non-meagre.*

A typical application of Baire's theorem:

1.67. Theorem. *$\mathcal{N} := \{f \in \mathcal{C}_{\mathbb{R}}([0,1]) : f$ is nowhere differentiable$\}$ is dense in $\mathcal{C}_{\mathbb{R}}([0,1])$ with respect to d_∞.*

sketch. Show $\mathcal{N}^c$ is meagre, then the claim follows from Corollary 66 iv). Observe $\mathcal{N}^c = \bigcup_{n\in\mathbb{N}} A_n^n$; where

$$A_n := \{f \in \mathcal{C}([0,1]) : \exists x \in [0,1] : \forall y \in [0,1] \text{ we have } |f(x) - f(y)| \leq n|x-y|\}$$

2 claims: First of all A_n is closed and A_n has empty interior (without proof). Thus A_n is nowhere dense and $\bigcup_{n\in\mathbb{N}} A_n$ is meagre, so by Lemma 64 iii) $\mathcal{N}^c$ is meagre. □

CHAPTER 2

Banach and Hilbert spaces

1. Vector spaces

CONVENTION. Let $X \neq \{0\}$ be a $\mathbb{K}$-vector space where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

2.1. DEFINITION. Let X be a vector space and $\varnothing \neq M \subseteq X$

- M is linearly independent $:\Leftrightarrow$ $\forall F \subseteq M$, $F \neq \varnothing$ and $|F| < \infty$ we have $\sum_{f\in F} \alpha_f f = 0$; $\alpha_f \in \mathbb{K} \Rightarrow \alpha_f = 0\ \forall f \in F$
- M is linearly dependent $:\Leftrightarrow$ M is not linearly independent.
- $B \subset X$ is a Hamel basis $:\Leftrightarrow$ B is linearly independent and every $x \in X$ can be written as a finite linear combination of elements of B ("B spans X")
- X has finite dimension $:\Leftrightarrow$ $\exists$ Hamel basis B of X with $|B| < \infty$
  $|B| \coloneqq \dim X$ is the dimension of X.
- X has infinite dimension $:\Leftrightarrow$ X does not have finite dimension.

2.2. REMARK. $\dim X$ is well defined, because it is independent of the choice of B (Theorem from Linear Algebra).

2.3. EXAMPLE. $l_C \coloneqq \{x = (x_j)_{j\in\mathbb{N}} \in l^p : x_j \neq 0$ for only finitely many $j \in \mathbb{N}\}$, where C is a compact subset (Note: $l_C \subset l^1$).
Let $e_n \coloneqq (0,..0,1,0,...,0)$ then $B \coloneqq \{e_n : n \in \mathbb{N}\}$ is a Hamel basis for l_C.
Note: B is not a Hamel basis for l^1. In fact l^1 though separable does not have a countable Hamel basis (Exercise). That is why the concept of a Hamel basis is not very useful.

2.4. THEOREM. *Every vector space $X \neq \{0\}$ has a Hamel basis.*

PROOF. Later (Uses Zorn's Lemma) □

2.5. COROLLARY. *X has infinite dimension $\coloneqq$:*

$$\forall n \in \mathbb{N} : \exists M_n \subset X : |M_n| = n \text{ and } M_n \text{ is linearly independent.}$$

PROOF. "$\Rightarrow$" By definition.
"$\Leftarrow$" Take a Hamel basis B of X(Thm. 4) with $|B| = \infty$ (otherwise X has finite dimension). Choose $M_n \subset B$ with $|M_n| = n$.
□

2.6. EXAMPLE. The following spaces are infinite dimensional: l^p $(\forall p \in [1,\infty])$, l_C, $\mathcal{C}(\overline{\Omega})$, for $\Omega \subset \mathbb{R}^d$ open and bounded.

Useful notion of basis allows infinite linear combinations, and thus requires convergence which implies a need for topology.

2. Banach Spaces

2.7. DEFINITION. X is a vector space, A mapping $\|\cdot\| \equiv \|\cdot\|_X : X \to [0,\infty[$ is a norm on X $:\Leftrightarrow$

1) definiteness: $\|x\| = 0 \Rightarrow x = 0$ (or $\|x\| > 0 \Rightarrow x \neq 0$)
2) pos. homogenity: $\|\alpha x\| = |\alpha|\,\|x\|$; $\forall x \in X, \alpha \in \mathbb{K}$

3) triangle ineq.: $\|x+y\| \leq \|x\| + \|y\|$; $\forall x, y \in X$

$(X, \|\cdot\|)$ is called a normed space. If only 2) and 3) hold, then $\|\cdot\|$ is a seminorm.

2.8. Remark. Let X be a normed space, then $d(x,y) \coloneqq \|x-y\| \quad \forall x,y \in X$ defines a metric on X and thus all topological notions from metric spaces apply to normed spaces. In particular $x \mapsto \|x\|$ $(= d(x,0))$ is continuous (see Cor. 1.22). A neighbourhood base of the norm topology at x is $\{B_{\frac{1}{k}}(x) : k \in \mathbb{N}\} = x + B_{\frac{1}{k}(0)}$. Here the addition of sets works like $A + B \coloneqq \{a+b : a \in A, b \in B\}$ and $a + B \coloneqq \{a\} + B$

Warning. Not every metric comes from a topology.

2.9. Example.
- $\forall p \in [1,\infty]$, $(l^p, \|\cdot\|)$ is a normed space.
- $\mathcal{C}_{\mathbb{K}}(X)$ for X a compact Hausdorff space, is a normed space with respect to $\|f\|_\infty \coloneqq \sup_{x\in X} |f(x)| \quad \forall f \in \mathcal{C}_{\mathbb{K}}$.
- $\mathbb{R}^n$, $\mathbb{C}^n$ with Euclidean norm.

2.10. Lemma. *The addition and scalar multiplication are continuous.*
- $x_k \xrightarrow{k\to\infty} x,\ y_k \xrightarrow{k\to\infty} y \Rightarrow x_k + y_k \xrightarrow{n\to\infty} x+y$
- $\alpha_k \xrightarrow{k\to\infty} \alpha,\ x_k \xrightarrow{k\to\infty} x \Rightarrow \alpha_k x_k \xrightarrow{k\to\infty} \alpha x$

Proof.
- $\|x_n + y_n - x - y\| \leq \|x_n - x\| + \|y_n - y\| \xrightarrow{k\to\infty} x+y.$
- $\|\alpha_n x_n - \alpha x\| \leq \underbrace{\|\alpha_n x_n - \alpha_n x\|}_{\underbrace{|\alpha_n|}_{\text{uniformly bounded in } \mathbb{K},\ \leq 2|\alpha|} \underbrace{\|x_n - x\|}_{\to 0}} + \underbrace{\|\alpha_n x - \alpha x\|}_{\underbrace{|\alpha_n - \alpha|}_{\to 0} \underbrace{\|x\|}_{<\infty}} \xrightarrow{k\to\infty} 0.$ ☐

2.11. Definition. A complete normed space is called a Banach space

2.12. Example.
- all examples from Example 9 are Banach spaces!
- $\mathcal{C}([0,1])$ with norm $\|f\| \coloneqq \int_0^1 |f(t)|\,dt$ is *not* a Banach space.

2.13. Theorem. *Every normed space X can be completed such that X is isometrically isomorphic to a dense linear subspace W of its completion, the Banach space $\overline{X}$, which is unique up to isometric isomorphism.*

Proof. analogous to the proof of Thm 1.3.2. Note that the isometry in that proof is a linear bijection (=: Homeomorphism) ☐

A more useful type of basis...

2.14. Definition. X a normed space. $\{e_n \in X : n \in \mathbb{N}\}$ is a Schauder basis in $X :\Leftrightarrow \forall x \in X\colon\ \exists_1 (x_n)_n \in \mathbb{K}^{\mathbb{N}} \colon \lim_{N\to\infty} \left\| x - \sum_{n=1}^N x_n e_n \right\| = 0$

Notation. $x = \sum_{n\in\mathbb{N}} x_n e_n$ *"infinite linear combinations", "convergent series"*

Example. Let $p \in [1,\infty[$ then $\{e_n : n \in \mathbb{N}\}$, $e_n \coloneqq (0,\cdots,0,1,0,\cdots)$ is a Schauder basis of l^p, because for $x = (x_1, x_2, \cdots) \in l^p$, $\|x\|_p < \infty \Rightarrow$

$$\forall \epsilon > 0\ \exists N\colon \sum_{n=N+1}^{\infty} |x_n|^p < \epsilon^p \Rightarrow \underbrace{\left\| x - \sum_{n=1}^N x_n e_n \right\|_p^p}_{(0,\ldots,0,x_{N+1},x_{N+2},\ldots)} < \epsilon^p$$

This fails for $p = \infty$!

2.15. Lemma. *If X is a normed space and has a Schauder basis e_n then it is separable.*

Proof. Let $\mathbb{K}_0 \coloneqq \begin{cases} \mathbb{Q}, & \mathbb{K} = \mathbb{R} \\ \mathbb{Q} + i\mathbb{Q}, & \mathbb{K} = \mathbb{C} \end{cases}$ $\quad A_n \coloneqq \left\{ \sum_{n=1}^N \alpha_n e_n : x_n \in \mathbb{K}_0 \right\} \subset X$ countable, $A = \bigcup_{n\in\mathbb{N}} A_n$ und $\overline{A} = X$, i.e. A is dense and a countable subset. ☐

2.16. REMARK. "$\Leftarrow$" does not hold in Lemma 15 (Enflo 1973)

2.17. LEMMA. *X is a Banach space and $A \subset X$ a linear subspace. Then*

$$A \text{ closed} \Leftrightarrow A \text{ complete.}$$

PROOF. See Lemma 1.29. □

2.18. THEOREM. *Let X be a normed space and $F \subset X$ a finite dimensional linear subspace of X. Then F is complete and closed.*

PROOF. Choose basis $\{e_1, \cdots, e_n\}$ in F (finite!) then $(F, \|\cdot\|)$ is isometric to $(\mathbb{K}^n, |||\cdot|||)$ with

$$|||\alpha||| := \left\| \sum_{\nu=1}^{n} \alpha_\nu e_n \right\| \qquad \forall \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{K}^n$$

a norm on $\mathbb{K}^n$ (check it!). But $\mathbb{K}^n$ is complete with respect to the Euclidean norm. Since all norms are equivalent on finite-dimensional spaces (Theorem from Linear Algebra) $\Rightarrow (\mathbb{K}^n, |||\cdot|||)$ complete $\Rightarrow (F, \|\cdot\|)$ complete $\Rightarrow F$ closed by Lemma 17 with $X = A = F$. □

As a preparation for a compactness result.

2.19. LEMMA (Riesz). *Let X be a Banach space, $U \subset X$ closed subspace $\Rightarrow \forall \lambda \in]0,1[\ \exists x_\lambda \in X \setminus U$ with $\|x_\lambda\| = 1$ and $\operatorname{dist}(x_\lambda, U) = \lambda$*

PROOF. Let $\lambda \in]0,1[$ (Exercise 7: $\forall x \in X \setminus U$, $\ d := \operatorname{dist}(x, U) > 0$). Since $\lambda < 1\ \exists u_\lambda \in U$: $d \le \|u_\lambda - x\| < \frac{d}{\lambda}$. Let $\gamma := \frac{1}{\|u_\lambda - x\|} \ge \frac{\lambda}{d}$ and $x_\lambda := \gamma(x - u_\lambda)$. Clearly $\|x_\lambda\| = 1$ and $x_\lambda \in X \setminus U$. Let $u \in U$

$$\Rightarrow \|x_\lambda - u\| = \|\gamma x - (u + \gamma u_\lambda)\| = \gamma \left\| x - \underbrace{\left(\frac{1}{\gamma} u + u_\lambda\right)}_{\in U} \right\| \ge \gamma d \ge \lambda. \qquad \square$$

Warning 1.43 illustrates the more general

2.20. THEOREM. *Let X be a normed space, then $\overline{B_1(0)}$ is compact if and only if $\dim X < \infty$.*

PROOF. "$\Leftarrow$" Exercise on the equivalence of norms. $\dim x < \infty$, hence X is isometrically isomorphic to $(\mathbb{K}^n, \|\cdot\|)$, ($\|\cdot\|$ is the standard norm). This implies the conclusion by Heine-Borel.

"$\Rightarrow$" Suppose $\dim X = \infty$. Pick $0 \ne x_1 \in X : \|x_1\| = 1 \left(x_1 \in \overline{B}_1(0)\right)$. Let $U_1 := \operatorname{span}\{x_1\}$; this is closed in X.Since U is closed we can apply Riesz for $\lambda := \frac{1}{2}$ and get the existence of $x_2 \in X \setminus U_1$ and $\|x_2\| = 1$ which implies $x_2 \in \overline{B}_1(0), \|x_2 - x_1\| \ge \frac{1}{2}$. Now let $U_2 := \operatorname{span}\{x_1, x_2\}\ldots$ choose $x_3\ldots$ Inductively we get a sequence $(x_n)_{n\in\mathbb{N}} \subset \overline{B}_1(0)$ with $\|x_n - x_m\| \ge \frac{1}{2}$; $\forall n \ne m$. $\Rightarrow (x_n)_n$ does not have a convergent subsequence and thus $\overline{B}_1(0)$ is not compact. □

3. Linear Operators

2.21. DEFINITION. Let X, Y be $\mathbb{K}$-vector spaces and $X_0 \subset X$ a subspace, then $T: X_0 \to Y$ is linear (a linear operator) $:\Leftrightarrow$

$$T(\alpha x + \alpha' x') = \alpha T(x) + \alpha' T(x') \qquad \forall x' x \in X_0,\ \forall \alpha, \alpha' \in \mathbb{K}$$

NOTATION. $Tx = T(x)$,

- $\operatorname{dom}(T) := X_0$ *domain of* T.
- $\operatorname{ran}(T) := T(X_0)$ *range of* T.
- $\ker(T) := \{X \in \operatorname{dom}(T) : Tx = 0\}$ *kernel of* T.

T is a (vector space) homeomorphism.

2.22. Example.
- Identity operator: $\mathrm{id}_X = \mathbb{1} = \mathrm{id}\colon X \to X, x \mapsto x$
- Let $X = Y = \mathcal{C}([0,1])$:
  - $T = \frac{\mathrm{d}}{\mathrm{d}x}$ with $\mathrm{dom}(T) = \mathcal{C}^1([0,1])$, $Tf \coloneqq f'$ differential operator.
  - $(Tf)(x) \coloneqq \int_0^x \mathrm{d}t f(t)$; $\forall x \in [0,1]$; $\forall f \in X = \mathrm{dom}(T)$ antiderivative.
  - $(Tf)(x) = xf(x)$; $\forall x \in [0,1]$; $\forall f \in X = \mathrm{dom}(T)$ multiplication operator.

2.23. Lemma. *Let T be a linear operator, then:*
- i) $\mathrm{ran}(T)$, $\ker(T)$ *are vector spaces*
- ii) $\dim(\mathrm{ran}(T)) \le \dim(\mathrm{dom}(T))$ [†]
- iii) $\ker(T) = \{0\}$ *if and only if there exists the inverse operator* $T^{-1}\colon \mathrm{ran}(T) \to \mathrm{dom}(T)$ *and* $T^{-1}T = \mathbb{1}_{\mathrm{dom}(T)}$

Proof. Linear algebra □

2.24. Remark.
- If T^{-1} exists, then it is a linear operator.
- The existence of T^{-1} does not imply $T \cdot T^{-1} = \mathbb{1}$, but only $T \cdot T^{-1} = \mathbb{1}_{\mathrm{ran}(T)}$, in other words $\ker(T) = \{0\} \not\Rightarrow T$ bijective.

Example. The shift operator T on l^∞, defined by $\forall x = (x_1, x_2, \dots)$

$$T(x)_n \coloneqq \begin{cases} 0, & n-1 \\ x_{n-1}, & n > 1 \end{cases}, \ \mathrm{dom}(T) = l^\infty,\ \mathrm{ran}(T) = \{y \in l^\infty : y_1 = 0\}$$

We now claim: $T^{-1}\colon (x_1, \cdots) \mapsto (x_2, \cdots)$, $\mathrm{dom}(T^{-1}) = l^\infty$, $T^{-1} \cdot T = \mathbb{1}_{l^\infty}$, but

$$TT^{-1}(x_1, x_2, \dots) = (0, x_1, x_2, \dots) \Rightarrow T \cdot T^{-1} = \mathbb{1}_{\mathrm{ran}\,T}$$

Note. Even though $\mathrm{dom}(T) = X$ and $\ker(T) = \{0\}$ we have $\mathrm{ran}(T) \subsetneq X$ (not possible if $\dim X < \infty$!)

2.25. Definition. Let X, Y be normed spaces, $T\colon \mathrm{dom}(T) \to Y$, then T is bounded $\Leftrightarrow$

$$\|T\| \equiv \|T\|_{\mathrm{dom}(T) \to Y} \coloneqq \sup_{\substack{x \in \mathrm{dom}\,T \\ x \neq 0}} \frac{\|Tx\|_Y}{\|x\|_Y} = \sup_{\substack{x \in \mathrm{dom}(T) \\ \|x\| = 1}} \|Tx\| < \infty$$

2.26. Example.
- $\mathbb{1}$ bounded with $\|\mathbb{1}\| = 1$
- $X = \mathcal{C}([a,b])$, $a < b$, $a, b \in \mathbb{R}$ with $\|\cdot\|_\infty$:
  - The derivative $T = \frac{\mathrm{d}}{\mathrm{d}x}$ on $\mathcal{C}^1([a,b]) \subseteq \mathcal{C}([a,b])$ (c.f. Example 22)). Then T is not bounded. Recall the definition of the norm:

    $$\|T\| \coloneqq \sum_{\substack{f \in \mathrm{dom}(T) \\ \|f\|_\infty = 1}} \|Tf\|_\infty \quad \text{with (here) } Tf = f'.$$

    If $0 \in]a,b[$, take $f_n(t) \coloneqq \sin(nt) \Rightarrow \|f_n\|_\infty \xrightarrow{n \to \infty} 1$. $f_n'(t) = n\cos(nt)$. $\Rightarrow \|f_n'\|_\infty = n \Rightarrow \|T\| \ge \|f_n'\|_\infty = n$; $\forall n \in \mathbb{N} \Rightarrow T$ unbounded. In general shift in such a way that the max of cos lies inside $]a,b[$
  - The antiderivative: choose $[a,b] = [0,1]$: $Tf(x) = \int_a^x f(t)\mathrm{d}t$ $\forall x \in [0,1]$. $\|Tf\|_\infty \le (1-0)\|f\|_\infty \le \|f\|_\infty$ $\forall f \in \mathcal{C}([0,1]) \Rightarrow \|T\| \le 1$
  - The multiplication operator

    $$(Tf)(t) \coloneqq tf(t) \forall t \in [a,b] \quad \text{with } \mathrm{dom}(T) = X$$

[†] "a linear operator preserves linear dependence"

Then $\|T\| = \sup_{f\in X} \frac{\|Tf\|_\infty}{\|f\|_\infty}$ and $\|Tf\|_\infty \leq \max\{|a|,|b|\}\cdot\sup|ft| \Rightarrow \|T\| \leq \max\{|a|,|b|\} \leq \infty$. Choose $f(t) = 1\ \forall t \in [a,b] \Rightarrow \|f\|_\infty = 1$ and $\|Tf\|_\infty = \sup_{t\in[a,b]}|t-1| = \max\{|a|,|b|\} \Rightarrow \|T\| = \max\{|a|,|b|\}$

2.27. Theorem. *Let X,Y be normed spaces, $T\colon \mathrm{dom}(T) \subseteq X \to Y$ a linear Operator. Then the following are equivalent:*

i) *T is continuous.*
ii) *T is continuous in some $x_0 \in \mathrm{dom}(T)$.*
iii) *There exists a constant c: $\|Tx\| \leq c\|x\|$, $\forall x \in \mathrm{dom}(T)$.*
iv) *T is bounded.*

Proof. **"*i)* ⇒ *ii)*":** clear

"*ii)* ⇒ *iii)*": We first prove that continuity in x_0 implies continuity in $0 \in X$. X is normed thus metric, hence first countable and therefore continuity is equivalent to sequential continuity. Let $(x_n)_{n\in\mathbb{N}} \subset \mathrm{dom}(T)\colon x_n \xrightarrow{n\to\infty} 0 \Rightarrow x_0 + x_n \xrightarrow{n\to\infty} x_0$ (not true in general, but in normed spaces all balls are the same). This implies

$$Tx_n + Tx_0 \overset{\text{linearity}}{=} T(x_n + x_0) \xrightarrow[\text{contnuity in } x_0]{n\to\infty} Tx_0$$

Now set $\epsilon = 1$ then there exists $\delta > 0$:

If $\tilde{x} \in \mathrm{dom}(T)$ with $\|\tilde{x}\| = \delta \Rightarrow \|T\tilde{x}\| \leq 1$ (continuous at 0)(∗).

For general $x \in \mathrm{dom}(T)$, $\|x\| \neq 0$ set $\tilde{x} := \frac{\delta}{\|x\|}x$, then

$$\tilde{x} \in \mathrm{dom}(T), \|\tilde{x}\| = \delta \overset{(*)}{\Longrightarrow} \frac{\delta}{\|x\|}\|Tx\| = \|T\tilde{x}\| < 1 \Rightarrow \|Tx\| < \frac{1}{\delta}\|x\|.$$

"*iii)* ⇒ *iv)*": clear, because $\frac{\|Tx\|}{\|x\|} < c$ for all $x \in \mathrm{dom}(T)$ with $x \neq 0$

$$\|T\| = \sup_{\substack{x\in\mathrm{dom}(T)\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = c.$$

"*iv)* ⇒ *i)*": Let $(x_n)_n \subset \mathrm{dom}(T)\colon x_n \xrightarrow{n\to\infty} x \in \mathrm{dom}(T)$, then

$$\|Tx_n - Tx\| = \|T(x_n - x)\| \leq \|T\| \underbrace{\|x_n - x\|}_{\xrightarrow{n\to\infty} 0} \Rightarrow \lim_{n\to\infty}\|Tx_n - Tx\| = 0.$$

Important Inequalities: $\|Ty\| = \left|\frac{Ty}{y}\right|\|y\| = \|T\|\,\|y\|$. □

2.28. Lemma. *Let T be a linear operator with* $\dim(\mathrm{dom}(T)) < \infty \Rightarrow T$ *is bounded.*

Proof. Lemma 23 ⇒ $\dim(\mathrm{ran}(T)) < \infty$. Same argument as in linear algebra (linear operator of finitedimensional spaces is bounded). □

2.29. Definition. Let X,Y be vector spaces, $T\colon \mathrm{dom}(T) \to Y$. Let $U \subseteq \mathrm{dom}(T)$ be a linear subspace and $\mathrm{dom}(T) \subseteq W$.

restriction of T to U: $T|_U\colon U \to Y$, $x \mapsto T|_U(x) = T(x)$.
extension of T to W: $\overline{T} : W \to Y, \overline{T}x \mapsto Tx,\ \forall x \in \mathrm{dom}(T)$, i.e. $T = \overline{T}|_{\mathrm{dom}(T)}$

(N.B.: restriction is linear, extension not necessarily)

2.30. Theorem (Bounded linear extension). *Let X be a normed space, Y a Banach space and $T : \mathrm{dom}(T) \subseteq X \to Y$ a bounded linear operator. Let Z be the completion of* $\mathrm{dom}(T)$. *Then there exists a linear extension $\widetilde{T} : Z \to Y$ with $\|\widetilde{T}\|_{Z\to Y} = \|T\|_{\mathrm{dom}(X)\to Y}$. If we identify $X \subseteq \widetilde{X}$ ($\widetilde{X}$: completion of X) and* $\mathrm{dom}(T) \subseteq Z \subseteq \widetilde{X}$. *Then the extension $\widetilde{T}$ is unique*

Note. i) If X is Banach, then uniqueness holds anyway
ii) $\overline{\mathrm{dom}(T)} \subseteq Z$, i.e. extension to closure as a special case

Proof. Let $x \in Z$. $\mathrm{dom}(T)$ dense in Z. $\Rightarrow \exists (x_n)_n \subset \mathrm{dom}(T) \colon x_n \xrightarrow{n\to\infty} x$ in $\widetilde{X}$ $\Rightarrow (x_n)_n$ is Cauchy. $\Rightarrow (Tx_n)_n \subset Y$ is Cauchy, because

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \le \|T\| \, \|x_n - x_m\| .$$

Y Banach then $\exists y \in Y \colon Tx_n \xrightarrow{n\to\infty} y$. Call $y =: \widetilde{T}x (\Rightarrow \widetilde{T}|_{\mathrm{dom}(T)} = T)$.

- $\widetilde{T}$ is well-defined (independent of sequence): Let $(x'n)_n$ be another sequence with the same properties as $(x_n)_n$ then

$$\|Tx_n - Tx'_m\| < \|T\| \, \|x_n - x'_m\| \xRightarrow{n,m\to\infty} \left\| \lim_{n\to\infty} Tx_n - \lim_{m\to\infty} Tx'_m \right\| = 0$$

$\Rightarrow y$ does not depend on the sequence (x_n) that converges to x.

- Linearity: Let $x, \tilde{x} \in Z$, $\alpha, \beta \in \mathbb{K}$

$$\Rightarrow \widetilde{T}(\alpha x + \beta \tilde{x}) = \lim_{n\to\infty} \Big(\underbrace{T(\alpha x_n + \beta \tilde{x}_n)}_{\alpha T x_n + \beta T \tilde{x}_n} \Big)$$

where $(\tilde{x}_n)_n \subset \mathrm{dom}(T)$ and $\tilde{x}_n \xrightarrow{n\to\infty} \tilde{x} \Rightarrow \widetilde{T}(\alpha x + \beta \tilde{x}) = \alpha \widetilde{T} x + \beta \widetilde{T} \tilde{x}$.

- Equality of norms: $\|\widetilde{T}\| \ge \|T\|$ clear, because $\widetilde{T}|_{\mathrm{dom}(T)} = T$.
- Continuity of $\|\cdot\|$: Let $x \in Z$, $(x_n) \subset \mathrm{dom}(T)$. $\lim_{n\to\infty} x_n = x$.

$$\|\widetilde{T}x\| \overset{(*)}{=} \lim \|Tx_n\| \le \lim_{n\to\infty} \|T\| \, \|x_n\| \le \|T\| \, \|x\|$$

$$\Rightarrow \frac{\|\widetilde{T}x\|}{\|x\|} \le T \quad \forall x \in \overline{\mathrm{dom}\, T} \Rightarrow \|\widetilde{T}\| \le \|T\|$$

- (*) is necessary to guarantee continuity in x!

□

2.31. Definition. For X, Y normed spaces, let

$$\mathrm{BL}_Y(x) \coloneqq \{T \colon X \to Y, T \text{ is bounded and linear}\}.$$

If $X = Y$ then $\mathrm{BL}_X(X) =: \mathrm{BL}(X)$.

2.32. Theorem. *Let X, Y be normed spaces, then* $(\mathrm{BL}_Y(X), \|\cdot\|_{X\to Y})$ *is a normed space. If Y is Banach, then so is* $\mathrm{BL}_Y(X)$.

Proof. $\mathrm{BL}_Y(X)$ is a vector space with zero element $0 \colon X \to X, x \mapsto 0$ and with respect to the operation

$$(\alpha T_1 + \beta T_2)(x) \coloneqq \alpha T_1 x + \beta T_2 x \quad \forall x \in X,\ \forall \alpha, \beta \in \mathbb{K},\ \forall T_1, T_2 \in \mathrm{BL}_X(X) \colon$$

- $\|\cdot\|$ is a norm on $\mathrm{BL}_Y(X)$:
  - $\|T\| = 0 \Rightarrow \|Tx\| = 0,\ \forall x \in X \Rightarrow Tx = 0,\ \forall x \in X \Rightarrow T = 0$
  - $\|(T_1 + T_2)x\| \le \|T_1 x\| + \|T_2 x\| \le (\|T_1\| + \|T_2\|) \|x\|$
    $\|T_1 + T_2\| \le \|T_1\| + \|T_2\|$ from $\triangle$-inequality of the norm
  - $\|\alpha T\| = \sup_{0 \ne x \in X} \frac{\|\alpha T x\|}{\|x\|} = |\alpha| \, \|T\|$

- Completeness if Y is Banach: Let $(T_n)_{n\in\mathbb{N}} \subset \mathrm{BL}_Y(X)$ be Cauchy: $\forall\epsilon > 0 \Rightarrow \exists N \in \mathbb{N}\ \forall k,l > N\ \|T_k - T_l\| \le \epsilon$.

$$\text{Since } \|T_k x - T_l x\| \le \|T_k - T_l\| \, \|x\| \quad \forall x \in X \tag{*}$$

we have that $(T_k x)_{k\in\mathbb{N}} \subset Y$ is Cauchy $\forall x \in X$. By the completeness of Y there exists $y_x \in Y : \lim_{k\to\infty} T_k x = y_x \coloneqq Tx$ this defines a mapping $T : X \to Y, T{:}\, X \to Y\ x \mapsto y_x$. Check:

T is linear: Let $x_1, x_2 \in X$ and $\alpha, \beta \in \mathbb{K}$:

$$\alpha T x_1 + \beta T x_2 = \lim_{k\to\infty} \underbrace{(\alpha T_k x_1 + \beta T_k x_2)}_{T_k(\alpha x_1 + \beta x_2)} = T(\alpha x_1 + \beta x_2)$$

***T* is bounded and the limit of T_n w.r.t. $\|\cdot\|$:** Let $\epsilon > 0$, then by (*) we know, that $\forall N \in \mathbb{N}\ \forall k,l > N,\ \forall x \in X$:

$$\|T_k x - T_l x\| \le \| \underbrace{T_k - T_l}_{<\epsilon} \| \, \|x\| \le \epsilon \|x\| .$$

Taking the limit of $l \to \infty$ we are given:

$$\|T_n x - Tx\| \le \epsilon \|x\| \Rightarrow \sup_{0\ne x\in X} \frac{\|(T_n - T)x\|}{\|x\|} \le \epsilon.$$

We have thus asserted that $T_n - T \in \mathrm{BL}_Y(X)$ and can use this now to show

$$T = \underbrace{T_n}_{\in \mathrm{BL}_Y(X)} - \underbrace{T_n - T}_{\in \mathrm{BL}_Y(X)} \in \mathrm{BL}_Y(X)$$

and the fact that $\|T_n - T\| < \epsilon$ gives us $T_k \xrightarrow{n\to\infty} T$ with respect to the operator norm.

□

4. Linear functionals and dual spaces

2.33. Definition. Let X be a normed space. A linear functional is a linear mapping $l : X \to \mathbb{K}$ with $\mathrm{dom}(l) \subseteq X$. The dual space is $X^* \coloneqq \mathrm{BL}_{\mathbb{K}}(X)$, i.e. the continuous linear functionals.

2.34. Corollary. *Let X be a normed space then X^* is complete (Banach Space)(no matter what X is) with respect to* $(\|\cdot\|_{X\to\mathbb{K}})$

Proof. $\mathbb{K}$ is complete and Theorem 32. □

2.35. Example. $X = \mathcal{C}([a,b]), a < b \in \mathbb{R}$ with $\|\cdot\|_n$

i) for $f \in X$ let $\mathfrak{I}(f) \coloneqq \int_a^b f(t)\mathrm{d}t$. This is clearly linear.

$$\mathfrak{I}(f) \le \int_a^b |f(t)|\,\mathrm{d}t \le (a-b)\,\|f\|_\infty$$

$$\Rightarrow \|\mathfrak{I}\|_{x\to\infty} = \sup_{0\ne f\in X} \frac{\|\mathfrak{I}(f)\|}{\|f\|_\infty} \le (b-a)\ (*) \Rightarrow \mathfrak{I} \in X^*$$

ii) Dirac δ–functional: for $f \in X$ and $t_0 \in]a,b[$ set $\delta_{t_0}(f) \coloneqq f(t_0)$. This is a linear functional. Mow $|\delta_{t_0}(f)| = |f(t_0)| \le \|f\|_\infty$ thus giving $\|\delta_{t_0}\|_{X\to\mathbb{C}} \le 1\ (**)$ and $\delta_{t_0} \in X^*$.

We have equalities at $(*)$ and $(**)$ (Choose $f(t) = 1\ \forall t$)

2.36. Theorem. *Let $p \in [1,\infty[$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $(l^p)^* \simeq l^q$ are isometrically isomorphic.*

PROOF. We only treat $p > 1$ (the case $p = 1$ is analogous). Let $f \in (l^p)^*$ and $x \in l^p$. $(e_n)_n = (\delta_{ij}) \in l^p$ is a Schauder basis of l^p (Remark 14'), thus we can decompose x like $(x_n)_{n\in\mathbb{N}} = x = \sum_{n=1}^{\infty} x_n e_n$. Since f is linear and continuous: $f(x) = \sum_{n=1}^{\infty} x_n f(e_n)$ (*). Now define $\tilde{x}_n$ for all $n, N \in \mathbb{N}$ like

$$\tilde{x}_n := \begin{cases} \frac{|f(e_n)|^q}{f(e_n)}, & f(e_n) \neq 0 \text{ and } n < N \\ 0, & \text{sonst} \end{cases} \quad \text{and } \tilde{x} := (\tilde{x}_n)$$

Then we can easily see: $0 \leq \sum_{n=1}^{N} |f(e_n)|^q = f(\tilde{x}) \leq \|f\|_{(l^p)^*} \|\tilde{x}\|_p$, but

$$\|\tilde{x}\|_p = \left(\sum_{n=1}^{N} |f(e_n)|^{\overbrace{(q-1)p}^{q}} \right) \Rightarrow \left(\sum_{n=1}^{N} (f(e_n))^q \right)^{\frac{1}{q}} \leq \|f\|_{l^{p*}} \quad \forall N \in \mathbb{N}$$

Hence $\mathfrak{I}\colon (l^p)^* \to l^q,\ f \mapsto (f(e_n))_{n\in N}$ is well defined and enjoys the properties

(1) $\mathfrak{I}$ is linear and $\|\mathfrak{I}f\|_q \leq \|f\|_{l^{p*}}$
(2) $\mathfrak{I}$ is onto, because if $y = (y_n)_n \in l^q$, then $f(e_n) := y$ and (*) define a linear functional f on X. Hölder $\Rightarrow |f(x)| \leq \|x\|_p \|y\|_q \Rightarrow f \in (l^p)^*$
(3) $\|l\|_{(l^p)^*} \leq \|\mathfrak{I}f\|_q$ (from (*) and Hölder)

Thus we have that $\mathfrak{I}$ is an isometric isomorphism. The case $p = 1$ works the same but instead of defining $\tilde{x}$, observe

$$|f(e_n)| \leq \|f\|_{(l^1)^*} \underbrace{\|e_n\|_1}_{=1} \quad \forall n \in \mathbb{N} \Rightarrow \|If\|_\infty \leq \|f\|_{(l^1)^*}$$ □

2.37. REMARK. (1) $c_0^* = l^1$
(2) Let X, Y be normed spaces with $X \subset Y : Y^* \subseteq X^*$
(3) $(l^\infty)^* \subset l^1$ (later we will see $\subsetneq$). Consider $l^1 \to (l^\infty)^*, x \mapsto f_x,\ f_x(y) = \sum_{n\in\mathbb{N}} x_n y_n \quad \forall y \in l^\infty$. This is well-defined (by Hölder), linear, isometric but not surjective. $(c_0,\ l^\infty,\ (l^\infty)^* \subseteq c_0^* = l^1)$

5. Hilbert Spaces

2.38. DEFINITION. Let X be a ($\mathbb{K}$-)vector space. A mapping $\langle\cdot,\cdot\rangle : X \to \mathbb{K}$ is an <u>inner product</u> (a scalar product) $\Leftrightarrow$ $\forall x, y, z \in X,\ \alpha, \beta \in \mathbb{K}$

i) $\langle x, x\rangle \geq 0$ if $\langle x, x\rangle = 0 \Rightarrow x = 0$ (non-degeneracy)
ii) $\langle x, \alpha y + \beta z\rangle = \alpha \langle x, y\rangle + \beta \langle x, z\rangle$ (linearity)
iii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ (conjugate symmetry)

$(X, \langle\cdot,\cdot\rangle)$ is an inner product space (or pre-Hilbert space)

2.39. LEMMA (Cauchy-Schwarz-(Bunjakowski-)Inequality).

$$\forall x, y \in X : |\langle x, y\rangle| \leq \langle x, x\rangle^{\frac{1}{2}} \langle y, y\rangle^{\frac{1}{2}}.$$

This turns into an equality $\Leftrightarrow$ x, y linearly independent.

PROOF. Linear Algebra. □

2.40. LEMMA. *Let X be an inner product space. Then X is a normed space with norm $\|x\| := \langle x, x\rangle^{\frac{1}{2}}$ $\forall x \in X$. Thus all notions of topological, metric and normed spaces apply to X. In particular $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{K}$ is continuous.*

PROOF. Definiteness and positive homogenity of $\|\cdot\|$ are clear. Triangle inequality:

$$\|x + y\|^2 = \|x\|^2 + 2\,\mathrm{Re} \underbrace{\langle x, y\rangle}_{\overset{CSI}{\leq} \|x\|\|y\|} + \|y\|^2 \leq (\|x\| + \|y\|)^2$$

Continuity of $\langle\cdot,\cdot\rangle$ follows from

$$|\langle x_n, y_m\rangle - \langle x, y\rangle| \le |\langle x_n - x, y_m\rangle| + |\langle x, y_m - y\rangle| \le \|x_n - x\| \, \|y_m\| + \|x\| \, \|y_m - y\| ,$$

continuity of $\|\cdot\|$ (Remark 8) and boundedness of $\sup_m \|y_m\| < \infty$ for converging sequences $x_n \to x,\ y_m \to y$. □

2.41. Definition. A complete inner product space is called a Hilbert Space.

2.42. Definition. A unitary operator is an isomorphism u with $\langle x, y\rangle_X = \langle ux, uy\rangle_H$

2.43. Theorem. *Every inner product space X can be completed: $\exists$ Hilbert spaces H and a unitary operator $U: X \to W$; $W \subset H$ being a dense subset. H is unique up to unitary copies.*

Proof. See Theorems 13 and 1.32. Define inner product on H by $\langle \tilde{x}, \tilde{y}\rangle_H \coloneqq \lim \langle x_n, y_n\rangle_X$ ($\tilde{x}, \tilde{y}$ are the equivalence classes of Cauchy sequences in X). Therefore we have shown $U: X \to W, x \mapsto [(x, x, \cdots)]$ to be unitary. □

2.44. Example.
- l^2 is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ with $\langle x, y\rangle = \sum_{n\in\mathbb{N}} \overline{x}_n y_n$ for $x = (x_n)_n,\ y = (y_n)_n$
- $\mathcal{C}([0,1])$ is an inner product space with respect to $\langle f, g\rangle \coloneqq \int_0^1 \mathrm{d}t \overline{f(t)} g(t)$ but not a Hilbert space (c.f. Example 1.24 ii))
- l^p for $p \neq 2$ is not an inner product space, because

2.45. Theorem (Jordan von Neumann). *Let X be a normed space. Then X is an inner product space $\Leftrightarrow$ $\|\cdot\|$ satisfies the parallelogram identity*

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) \ \forall x, y \in X$$

Proof. "$\Rightarrow$" Elementary computation
"$\Leftarrow$" By polarization. Define

$$\langle x, y\rangle \coloneqq \begin{cases} \frac{1}{2}\left(\|x+y\|^2 - \|x-y\|^2\right) & \mathbb{K} = \mathbb{R} \\ \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + \|x+iy\|^2 - \|x-iy\|^2\right) & \mathbb{K} = \mathbb{C} \end{cases}$$

Symmetry and definiteness obvious; bi-/sesquilinearity is clear from the parallelogram identity: (here only $\mathbb{K} = \mathbb{R}$)

$$\begin{aligned} \langle x, y\rangle + \langle x, z\rangle &= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + \|x+z\|^2 - \|x-z\|^2\right) \\ &= \frac{1}{2}\left(\left\|x + \frac{y+z}{2}\right\|^2 - \left\|x - \frac{y+z}{2}\right\|^2\right) \\ &= 2\left\langle x, \frac{y+z}{2}\right\rangle && 1) \\ x = 0 \text{ in } (*) &\Rightarrow \langle x, y\rangle = 2\langle x, y/2\rangle && 2) \\ 1) \text{ and } 2) \ &\Rightarrow \langle x, y\rangle + \langle x, z\rangle = \langle x, y+z\rangle && 3) \\ \text{by induction from } 3) &: \langle x, ny\rangle = n\langle x, y\rangle \quad \forall n \in \mathbb{N}_0 && 4) \end{aligned}$$

$z = -y$ in 3) $\Rightarrow \langle x, y\rangle = -\langle x, -y\rangle \Rightarrow$ 4) holds for all $n \in \mathbb{Z}_0$.
Let $m \in \mathbb{Z}\setminus\{0\}$ 4) with $\frac{y}{m}$ instead of y:

$$\left\langle x, \frac{n}{m} y\right\rangle = \frac{n}{m}\cdot m\left\langle x, \frac{y}{m}\right\rangle = \frac{n}{m}\langle x, y\rangle$$

This holds for rational n and thus for all $n \in \mathbb{R}$ by continuity. □

2.46. Definition. Let X be a inner product space:

- $x, y \in X$ orthogonal $\Leftrightarrow \langle x, y\rangle = 0$ (in symbols $x \perp y$)
- Let $A \subseteq X$ be a subspace. The orthogonal complement of A is defined as

$$A^\perp := \{x \in X;\ x \perp a,\ \forall a \in A\}$$

- A set $\{e_\alpha \in X : \alpha \in I\}$ is an orthonormal set $:\Leftrightarrow$

$$\|e_\alpha\| = 1 \quad \forall \alpha \in I \text{ and } \langle e_\alpha, e_\beta\rangle = 0 \quad \forall \alpha, \beta$$

- an orthonormal set $\{e_\alpha\}$ is an orthonormal basis (ONB) (or a complete orthonormal set) $:\Leftrightarrow \langle e_\alpha, x\rangle = 0 \quad \forall \alpha \in I \Rightarrow x = 0$, i.e. the zero vector is the only one which is orthogonal to all the e_n's

2.47. Lemma. (1) *$A^\perp$ is closed.*
(2) *An orthonormal set is linearly independent.*

Proof. i) Exercise
ii) Let $N \in \mathbb{N}$ and $\alpha_i \in I$ pairwise distinct. If $0 = \sum_{n=1}^N \lambda_n e_{\alpha_n}$, $\lambda_n \in \mathbb{K}$ then

$$0 = \left\langle e_{\alpha_m}, \sum_{n=1}^N \lambda_n e_{\alpha_n}\right\rangle = \sum_{n=1}^N \lambda_n \underbrace{\left\langle e_{\alpha_m}, e_{\alpha_n}\right\rangle}_{\delta_{mn}} = \lambda_m,\ \forall m \in \{1, \cdots, N\}$$ □

2.48. Theorem. *Let X be an inner product space and $\{e_\alpha\}_{\alpha \in I}$ an orthonormal set, then $\forall x \in X,\ \forall J \in I, |J| < \infty$*

$$\|x\|^2 = \sum_{\alpha \in J} |\langle e_\alpha, x\rangle|^2 + \left\| x - \sum_{\alpha \in J} \langle e_\alpha, x\rangle e_\alpha \right\|^2.$$

In particular:

$$\|x\|^2 \geq \sum_{\alpha \in I} |\langle e_\alpha, x\rangle|^2 \ \textit{(Bessel's Inequality).}$$

If X is a Hilbert space and $\{e_\alpha\}_\alpha$ an ONB, then

$$x = \sum_{\alpha \in I} \langle e_\alpha, x\rangle e_\alpha, \quad \|x\|^2 = \sum_{\alpha \in I} |\langle e_\alpha, x\rangle|^2 \qquad \textit{(Parseval's equality)}$$

Digression. Uncountable series:
If $\sum_{\alpha \in I} c_\alpha < \infty \Rightarrow c_\alpha = 0$ except for countably many α! Let $S_0 =: c_\alpha \geq 1\}$, $S_n := \{\alpha \in I : \frac{1}{n} > c_\alpha \geq \frac{1}{n+1}\}$. If for some $n \in \mathbb{N} : |S_n| = \infty$ then $\sum_{\alpha \in I} c_\alpha = \infty$ and thus $\{\alpha \in I : c_\alpha > 0\} = \bigcup_{n \in \mathbb{N}} S_n$ is countable

Proof. Let $x = \underbrace{\sum_{\alpha \in J} \langle e_\alpha, x\rangle e_\alpha}_{=:u} + \underbrace{x - \sum_{\alpha \in J} \langle e_\alpha, x\rangle e_\alpha}_{=:v}$. Now:

$$\langle u, v\rangle = \left\langle \sum_{\alpha \in J} \langle e_\alpha, x\rangle e_\alpha, \sum_{\beta \in J} \langle e_\beta, x\rangle e_\beta \right\rangle = \sum_{\alpha, \beta \in J} \overline{\langle e_\alpha, x\rangle} \langle e_\alpha, x\rangle \langle e_\alpha, e_\beta\rangle$$
$$= \sum_{\alpha \in J} |\langle e_\alpha, x\rangle|^2$$

and $\langle u, v\rangle = \langle u, x\rangle - \langle u, u\rangle = 0$ since

$$\langle u, x\rangle = \left\langle \sum_{\alpha \in J} \langle e_\alpha, x\rangle e_\alpha, x \right\rangle = \sum_{\alpha \in J} \overline{\langle e_\beta, x\rangle} \langle e_\alpha, x\rangle$$

$$\Rightarrow \|x\|^2 = \langle x, x\rangle = \|u\|^2 + \|v\|^2 \ \text{(and neglect } \|v\|^2\text{)}$$

$$\|x\|^2 \geq \sup_{\substack{J \leq I \\ |J| < \infty}} \sum_{\alpha \in J} |\langle e_\alpha, x\rangle|^2 =: \sum_{\alpha \in I} |\langle e_\alpha, x\rangle|^2 \text{ which is an uncountable series}$$

Fix $x \in X \xRightarrow{\text{Bessel}} \exists (\alpha_n)_{n\in\mathbb{N}} \subset I$:
$\sum_{\alpha\in I} |\langle e_\alpha, x\rangle|^2 = \sum_{n\in\mathbb{N}} |\langle e_{\alpha_n}, x\rangle|^2$ (finite!) Let $x := \sum_{\nu=1}^{n} \langle e_{\alpha_\nu}, x\rangle e_{\alpha_\nu}$ for $n \in \mathbb{N} \Rightarrow (x_n)_n \subset X$ is Cauchy, because for $n > m$:

$$\|x_n - x_m\|^2 = \left\| \sum_{\nu=m+1}^{n} \langle e_{\alpha_\nu}, x\rangle e_{\alpha_\nu} \right\|^2 = \sum_{\nu=m+1}^{n} |\langle e_{\alpha_\nu}, x\rangle|^2$$

$\Rightarrow \exists x' \in H$ completion of $x' = \lim_{n\to\infty} x_n$ Now:

$$\langle e_{\alpha_m}, x - x'\rangle = \langle e_{\alpha_m}, x\rangle - \lim_{n\to\infty} \sum_{\nu=1}^{n} \langle e_{\alpha_\nu}, x\rangle \underbrace{\langle e_{\alpha_m}, e_{\alpha_n}\rangle}_{\delta_{m\nu}} = 0 \quad \forall m \in \mathbb{N}$$

and if $\alpha \neq \alpha_n \quad \forall n \in \mathbb{N}$:

$$\langle e_\alpha, x - x'\rangle = \underbrace{\langle e_\alpha, x\rangle}_{=0} - \lim_{n\to\infty} \sum_{\nu=1}^{n} \langle e_{\alpha_\nu}, x\rangle \underbrace{\langle e_\alpha, e_{\alpha_0}\rangle}_{0} = 0$$

$\Rightarrow (x - x') \perp e_\alpha \quad \forall \alpha \in \mathfrak{I} \xRightarrow{\text{ONB}} x = x'$ and

$$\|x\|^2 = \lim_{n\to\infty} \|x_n\|^2 = \sum_{\nu\in\mathbb{N}} |\langle e_{\alpha_\nu}, x\rangle|^2 = \sum_{\alpha\in\mathfrak{I}} |\langle e_\alpha, x\rangle|^2 .$$ □

2.49. Corollary. *Let X be a Hilbert space and $\{e_\alpha\}_{\alpha\in\mathfrak{I}}$ an ONB. If $\sum_{\alpha\in\mathfrak{I}} |c_\alpha|^2 < \infty$ for $\{c_\alpha : \alpha \in \mathfrak{I}\} \subset \mathbb{K}$ then $\exists x \in X : x = \sum_{\alpha\in\mathfrak{I}} c_\alpha e_\alpha$*

2.50. Theorem. *Every Hilbert Space X has an ONB. Moreover X is separable if and only if X has a countable ONB.*

Proof. "$\Leftarrow$" Let X have a countable ONB then by Thm 47 it has a Schauder basis. Now apply Lemma 15.

"$\Rightarrow$" Let $\{x_n \in X \setminus \{0\} : n \in \mathbb{N}\}$ be dense. Apply Gram-Schmidt: $e_1 := \frac{x_1}{\|x_1\|}$. Throw away all x_n, $n \geq 1$ which are linearly dependent on e_1. Let x_1 be the smallest remaining index.
$\tilde{e}_2 := x_{n_1} - \langle e_1, x_{n_1}\rangle e_1 \Rightarrow \tilde{e}_2 \perp e_1$ and set $e_2 := \frac{\tilde{e}_2}{\|\tilde{e}_2\|}$. Throw away all $x_n, n \geq n_1$ ($x_n \in \operatorname{span}(e_1, e_2)$)
$\vdots$
$\Rightarrow \{e_n\}_n$ is an orthogonal set and every x_n a finite linear combination of the e_n's $\Rightarrow \operatorname{span}\{e_n : n\}$ dense in X.
ONB: Let $y \perp e_n \quad \forall n$ but: $\exists (y_n)_n \subset \operatorname{span}\{e_n\}$: $y_n \xrightarrow{n\to\infty} y$. Since $\langle y, y_n\rangle = 0 \quad \forall n \in \mathbb{N} \Rightarrow \|y\|^2 = \lim_{n\to\infty} \langle y, y_n\rangle = 0 \Rightarrow y = 0$. Existence of an ONB in a non-separable space use Zorn's Lemma, which we will complete later. □

2.51. Theorem. *Let X be a Hilbert space of finite dimension $N \in \mathbb{N}$, then $X \simeq \mathbb{C}^N$. If X is infinite dimensional and separable then $X \simeq l^2$*

Proof. 2nd. case: Let $\{e_n\}_{n\in\mathbb{N}}$ be an ONB. Define a linear operator:
$U : X \to l^2, x \mapsto (\langle e_n, x\rangle)_{n\in\mathbb{N}}$ surjective by Cor. 48. By Parseval we have welldefinedness $\forall x \in X$ and $\|x\|_X^2 = \|Ux\|_{l^2} = \sum_{n\in\mathbb{N}} |\langle e_n, x\rangle|^2 \Rightarrow \langle x, y\rangle = \langle U_x, U_y\rangle_{l^2}$ by polarization therefore U is unitary. The case of an Euclidean X of infinite dimension works analogously. □

2.52. Definition (Direct sum). Let $(X_1, \langle\cdot,\cdot\rangle_1)$, $(X_2, \langle\cdot,\cdot\rangle_2)$ be inner product spaces, then

$$X_1 \oplus X_2 := \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_j \in X_j, j = 1, 2 \right\}$$

is an inner product space with respect to the scalar product.

$$\left\langle\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\rangle_{\oplus} := \langle x_1, y_1\rangle_1 + \langle x_2, y_2\rangle_2$$

If X_1, X_2 are Hilbert spaces, then so is $X_1 \oplus X_2$

2.53. Example. i) Vector valued functions

$$C_1([0,1]) \oplus C_1([0,1]) = \left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : [0,1] \to \mathbb{C}^2 : f_j \text{ continuous} \right\}$$

with $\langle f, g\rangle_{\oplus} = \int_0^1 \mathrm{d}t[\overline{f_1(t)}g_1(t) + \overline{f_2(t)}g_2(t)]$

ii) orthogonal decomposition:
Let X be a Hilbert space and $A \subset X$ a closed subspace, then A is a Hilbert space (with respect to the inherited $\langle\cdot,\cdot\rangle$) so is $A^\perp$(by Lemma 46 (i))

$$A \oplus A^\perp = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \in A, y \in A^\perp \right\}$$

$$\left\langle\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}\right\rangle_{\oplus} = \langle x, x'\rangle_A + \langle y, y'\rangle_{A'} \overset{A \perp A'}{=} \langle x + y, x' + y'\rangle_X$$

$\Rightarrow A \oplus A^\perp \simeq A + A^\perp = \{x + y \in X : x \in A, y \in A^\perp\} = X$ because:

2.54. Theorem (projection theorem). *Let X be a Hilbert space and $A \subset X$ be a closed subspace, then $\forall x \in X$, $\exists_1 z \in A$, $\exists_1 w \in A^\perp : x = z + w$*

2.55. Lemma. $\forall x \in X\ \exists_1 z \in A : \|x - z\| = \operatorname{dist}(x, A)$, *$z$ is then called the closest element.*

Proof of Lemma. Let $d := \operatorname{dist}(x, A) \Rightarrow \exists (y_n)_n \subset A : \lim_{n\to\infty} \|x - y_n\| = d$. Now

$$\begin{aligned}
\|y_n - y_m\|^2 &= \|y_n - x + x - y_m\|^2 \\
&= 2\left(\|y_n - x\|^2 + \|y_m - x\|^2\right) - \underbrace{\|2x - y_n - y_m\|^2}_{4\|x - \frac{1}{2}(y_n + y_m)\| \geq 4d^2} \\
&\leq 2\left(\underbrace{\|y_n - x\|}_{n \to d^2} + \underbrace{\|y_m - x\|}_{m \to d^2}\right) - 4d^2
\end{aligned}$$

Thus $(y_n)_n$ is Cauchy in A and since A is complete we know $y_n \xrightarrow{n\to\infty} z \in A \Rightarrow \|x - z\| = d$. Moreover $z \in A$ unique, for assume $\exists (y'_m) \subset A : \|y'_m - x\| \to d \Rightarrow y'_m \xrightarrow{m\to\infty} z' \in A$ then

$$\|z - z'\|^2 = \lim_{\substack{n\to\infty \\ m\to\infty}} \|y_m - y'_m\|^2 \leq \lim_{\substack{n\to\infty \\ m\to\infty}} 2\left(\|y_n - x\|^2 + \|y'_m - x\|^2\right) - 4d = 0 \Rightarrow z = z' \qquad \square$$

Proof of Theorem. For $x \in X$ let z be as in the Lemma and set $w := x - z$. Then for $t \in \mathbb{R}$ $\forall y \in A, y \neq 0$

$$d^2 \leq \|x - (z + ty)\|^2 = \|w - ty\|^2 = d^2 - 2t \operatorname{Re}\langle w, y\rangle + t^2 \|y\|^2$$

$$\Rightarrow t^2 - 2\frac{\operatorname{Re}\langle w, v\rangle}{\|y\|^2} \geq 0,\ \forall t \in \mathbb{R} \quad \Rightarrow \operatorname{Re}\langle w, y\rangle = 0\ \forall y \in A$$

Repeat argument with $t \to it \Rightarrow \operatorname{Im}\langle w, y\rangle = 0,\ \forall y \in A \Rightarrow w \in A^\perp$. Uniqueness of decomposition at

$$x = \underbrace{z}_{\in A} + \underbrace{w}_{\in A^\perp} = \underbrace{z'}_{\in A} + \underbrace{w'}_{\in A^\perp} \Rightarrow \underbrace{z - z'}_{\in A} = \underbrace{w' - w}_{\in A^\perp} \in A \cap A^\perp = \{0\} \qquad \square$$

A Hilbert space is canonically isomorphic to its dual:

2.56. THEOREM (Riesz representation). *Let X be a Hilbert space and $l \in X^*$. Then $\exists_1 y_l \in X$: $l(x) = \langle y_l, x\rangle$, $\forall x \in X$ and $\|l\|_{X^*} = \|y_l\|_X$*

PROOF. $\ker(l)$ is a closed subspace in X (let $(x_n)_n \subset \ker(l), x_n \to x$, by continuity of l: $l(x) = \lim_{n\to\infty} l(x_n) = 0$). If $\ker(l) = X \Rightarrow l = 0$ and $y_l = 0 \in X$ is the only choice. So assume $\ker(l) \subsetneq X$. Thus by completeness of X and Thm. 53 we reach $\ker(l)^\perp \supsetneq \{0\}$. So let $0 \neq x_0 \in \ker(l)^\perp$ and set $y_l \coloneqq \frac{\overline{l(x_0)}}{\|x_0\|^2} x_0$.
Clearly if $x \in \ker(l) \Rightarrow \langle y_l, x\rangle = 0 = l(x)$. Now let $x = \alpha x_0, \alpha \in \mathbb{K} \Rightarrow l(x) = \alpha l(x_0) = \langle y_l, x\rangle$. Linearity $\Rightarrow l$ and $\langle y_l, \cdot\rangle$ agree on $\operatorname{span}(\ker(l), x_0)$, but $\operatorname{span}(\ker(l), x_0) = X$ because

$$\forall x \in X : x = \underbrace{\left(x - \frac{l(x)}{l(x_0)} x_0\right)}_{\in \ker(l)} + \frac{l(x)}{l(x_0)} x_0 \Rightarrow l = \langle y_l, \cdot\rangle \in X$$

<u>Uniqueness</u>: assume $\exists y' \in X$: $\langle y_l, \cdot\rangle = \langle y', \cdot\rangle \Rightarrow \langle y_l - y', , \rangle\, x = 0$, $\forall x \in X$, set $x = y_l - y'$, then $\|y_l - y'\|^2 = 0$ and hence $y_l = y'$
<u>Norm</u>: Without loss of generality $\forall l \neq 0$

$$\|l\|_{X^*} = \sup_{0\neq x\in X} \frac{|l(x)|}{\|x\|_X} = \sup_{0\neq x\in X} \frac{|\langle y_l, x\rangle|}{\|x\|_X} \qquad \|l\|_{X^*} \geq \frac{l(y_l)}{\|y_l\|_X} = \frac{\langle y_l, y_l\rangle}{\|y_l\|_X} = \|y_l\|_X \qquad \square$$

2.57. COROLLARY. *Let X be a Hilbert space then the mapping $\mathcal{A} : X^* \to X, l \mapsto y_l$ (like in 55) is an antilinear, bijective isometry.*

PROOF. Let $l_1, l_2 \in X^*$ $\alpha, \beta \in \mathbb{K}$

$$\Rightarrow \alpha l_1 + \beta l_2 = \alpha \langle y_{l_1}, \cdot\rangle + \beta \langle y_{l_2}, \cdot\rangle = \left\langle \overline{\alpha} y_{l_1} + \overline{\beta} y_{l_2}, \cdot\right\rangle$$
$$\Rightarrow \mathcal{A}(\alpha l_1 + \beta l_2) = \overline{\alpha}\mathcal{A}(l_1) + \overline{\beta}\mathcal{A}(l_2)$$

antilinear isometry from Thm 55 (and thus injective)
surjectiveness: Let $y \in X$, then $x \mapsto \langle y, x\rangle \in X^*$ is linear and bounded:

$$\sup_{0\neq x\in X} \frac{|\langle y, x\rangle|}{\|x\|} \overset{\text{CSI}}{\leq} \|y\| < \infty \qquad \square$$

CHAPTER 3

Measures, Integration and L^p-spaces

1. Measures

Measures cannot be defined satisfactorily on the power set, thus:

3.1. Definition. Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$, $\mathcal{A}$ is a σ-algebra (σ-field) :⇔

- $X \in \mathcal{A}$
- $A \in \mathcal{A} \Rightarrow A^c \subset \mathcal{A}$
- $A_n \in \mathcal{A},\ \forall n \in \mathbb{N} \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$

thus $\emptyset \in \mathcal{A}$, $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$; $A \setminus B \in \mathcal{A}$ if $A, B \in \mathcal{A}$. We say A measurable ⇔ $A \in \mathcal{A}$

3.2. Lemma. *Let $\mathcal{G} \subseteq \mathcal{P}(X)$, then there exists a smallest σ-algebra $\sigma(\mathcal{G})$: $\mathcal{G} \subseteq \sigma(\mathcal{G})$ (σ-algebra generated by $\mathcal{G}$)*

3.3. Definition. Let $(X, \mathcal{T})$ be a topological space then $\mathcal{B} \equiv \mathcal{B}(X) \coloneqq \sigma(\mathcal{T})$ Borel σ-Algebra

3.4. Definition.
i) Let $\mathcal{A}$ be a σ-Algebra on X. A mapping $\mu : \mathcal{A} \to [0, \infty]$ is a (positive) measure :⇔
- $\mu(\emptyset) = 0$
- $A_n \in \mathcal{A}$; $n \in \mathbb{N}$ pairwise disjoint $\Rightarrow \mu(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n)$ (σ-additivity)

$(X, \mathcal{A}, \mu)$ is called a measure space.

ii) If $\mu(X) = 1$ then μ is called a probability measure,
a finite measure, if $\mu(X) < \infty$ and
a σ-finite measure, if $X = \bigcup_n A_n$, $A_n \in \mathcal{A}$, $\forall n \in \mathbb{N}$ and $\mu(A_n) < \infty$.

iii) If X is Hausdorff, $\mathcal{A} = \mathcal{B}(X)$ and $\mu(K) < \infty$ for all compacts sets $K \subseteq X$ then μ is a Borel measure.

3.5. Definition.
i) $\mathcal{H} \subseteq \mathcal{P}(X)$ is a semi-ring ⇔
- $\emptyset \in \mathcal{H}$
- $A, B \in \mathcal{H} \Rightarrow A \cap B \in \mathcal{H}$
- $A, B \in \mathcal{H} \Rightarrow \exists c_1, \cdots, c_n \in \mathcal{H}\ c_i \neq c_j : A \setminus B = \bigcup_{k=1}^n c_k$

ii) A mapping $\mu : \mathcal{H} \to [0, \infty]$ is a premeasure :⇔
- $\mu(\emptyset) = 0$
- $A_n \in \mathcal{H}$, $\forall n \in \mathbb{N}$, $A_i \neq A_j$ and $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{H}$

$$\Rightarrow \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

3.6. Theorem (Kolmogorov extension of pre-measures).
Let μ be a σ-finite pre-measure on a semi-ring $\mathcal{H}$, then it has a unique extension to a measure on $\sigma(\mathcal{H})$.

3.7. Example. $X = \mathbb{R}$, $\mathcal{H} \coloneqq \{]a, b] : a, b \in \mathbb{R}, a \leq b\}$ is a semi-ring.

- $\mu(]a, b]) \coloneqq b - a$ defines a σ-finite pre-measure on $\mathcal{H}$
- $\sigma(\mathcal{H}) = \mathcal{B}(\mathbb{R})$

⇒ ∃ extension λ of μ to $\mathcal{B}(\mathbb{R})$: λ is called the Lebesgue-Borel measure (it is a Borel measure in the sense of 4 *iii*)).

3.8. Definition. $(X, \mathcal{A}, \mu)$ a measure space

- $A \subseteq X$ is a μ-null set $\Leftrightarrow A \in \mathcal{A}$ and $\mu(A) = 0$.
- a statement $p(x)$ holds for μ-almost all $x \in X$: $\Leftrightarrow \exists$ a μ-null set $\mathcal{N} \subset X$: $P(x)$ holds $\forall x \in X \setminus \mathcal{N}$
- $(X, \mathcal{A}, \mu)$ is complete $:\Leftrightarrow$ if A is a μ-null set and $B \subset A$, then so is B. (crucial $B \in \mathcal{A}$!)

3.9. Theorem (Completion). *Every measure space $(X, \mathcal{A}, \mu)$ has a completion $(X, \mathcal{A}^*, \mu^*)$ that is $C \in \mathcal{A}^* \Leftrightarrow \exists A, B \in \mathcal{A} : A \subseteq C \subseteq B$ and $\mu(A) = \mu(B) =: \mu^*(C)$*

3.10. Definition (Lebesgue measure). The completion of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ is the $(\mathbb{R}, \mathcal{B}(\mathbb{R})^*, \lambda^*)$ where λ^* is the Lebesgue measure.

3.11. Definition. Let $\mathcal{A}^{(1)}$ be a σ-algebra in $X^{(1)}$ and $f : X \to X'$, then f is $(\mathcal{A}, \mathcal{A}')$-measurable $:\Leftrightarrow f^{-1}(A') \in \mathcal{A},\ \forall A' \in \mathcal{A}'$

3.12. Lemma. *If $\mathcal{A}' = \sigma(\mathcal{G}')$, then $f^{-1}(A') \in \mathcal{A}' \in \mathcal{G}' \Rightarrow f$ is $\mathcal{A}$-$\mathcal{A}'$-measurable.*

3.13. Corollary. *(X, τ) and (X', τ') topological spaces and $f : X \to X'$ continuous $\Rightarrow f$ $\mathcal{B}(X)$-$\mathcal{B}(X')$-measurable.*

3.14. Theorem. *Let $f, g, f_n : X \to \mathbb{K}$ be measurable then the following are measurable too:* $\operatorname{Re} f, \operatorname{Im} f, f + g, f \cdot g, \sup f_n, \inf f_n, \limsup f_n, \liminf f_n, \lim_n f_n$ *(if it exists), also true for $\mathbb{K} = \overline{\mathbb{R}}$*

2. Integration

3.15. Definition. Let $(X, \mathcal{A}, \mu)$ be a measure space.

i) $f : X \to [0, \infty[$ is a simple function (step function) $:\Leftrightarrow \exists N \in \mathbb{N},\ \alpha_1, \cdots, \alpha_N > 0$ and $A_1, \cdots, A_N \in \mathcal{A}$:

$$f = \sum_{n=1}^{N} \alpha_n \mathbb{1}_{A_N}, \quad \text{where } \mathbb{1}_A(x) := \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

is the indicator function ($\mathbb{1}_A$ measurable $\Leftrightarrow A$ measurable) (Without loss of generality one can assume that $\alpha_1, \cdots, \alpha_N$ and $A_1, \ldots, A_N$ are pairwise disjoint)

ii) (μ-) integral of a single function: $\int_X \mu(\mathrm{d}x) f(x) := \sum_{n=1}^{N} \alpha_N \mu(A_n)$ (can be $+\infty$) other notations: $\int_X \mathrm{d}\mu(x) f(x) = \int_X \mathrm{d}\mu f = \int_X f(x) \mathrm{d}\mu(x)$

3.16. Lemma. *Let $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, $f \geq 0$ then*

$$\begin{matrix} f \text{ is } (\mathcal{A}\text{-}\overline{\mathcal{B}}\text{-}) \text{ measurable} \\ (\overline{\mathcal{B}} = \sigma(\mathcal{B} \cup \{+\infty\} \cup \{-\infty\}) \end{matrix} \Leftrightarrow \begin{cases} \exists (f_n) \text{ sequence of simple functions} \\ X \to [0, \infty[\text{ with } f_n \leq f_{n-1}\ \forall n \text{ and} \\ f = \lim_{n\to\infty} f_n = \sup_n f_n \end{cases}$$

3.17. Definition. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to \overline{\mathbb{R}}$.

i) If $f \geq 0$ measurable: (μ-) integral of f:

$$\int_X \mathrm{d}_\mu f := \lim_{n\to\infty} \int_X \mathrm{d}_\mu f_n$$

(where f_n are a monotonous sequence of functions, c.f. Lemma 16)

ii) Let $f_+ := \max\{f, 0\}$, $f_- := \max\{0, -f\}$, $f = f_+ - f_-$ (decomposition into negative and positive parts)

$$f \text{ (μ-) integrable} :\Leftrightarrow f \text{ measurable and } \int_X \mathrm{d}_\mu f_\pm < \infty.$$

In this case: $\int_X \mathrm{d}_\mu f := \mathrm{d}_\mu f_+ - \int_X \mathrm{d}_\mu f_-$.

3.18. REMARK. i) Definition 17 i) does not depend on the choice of $(f_n)_n$

ii) If $A \subseteq X$ measurable $\int_A \mathrm{d}_\mu f \coloneqq \int_X \mathrm{d}_\mu f \mathbb{1}_A$

$$\text{(in particular } f = 1\text{)}: \int_A \mathrm{d}_\mu = \int_X \mathrm{d}_\mu \mathbb{1}_A = \mu(A).$$

iii) $f: X \to \mathbb{C}$ measurable: f integrable $\Leftrightarrow$ $\operatorname{Re} f, \operatorname{Im} f$ integrable. In this case

$$\int_X \mathrm{d}_\mu f \coloneqq \int_X \mathrm{d}_\mu \operatorname{Re} f + i \int_X \mathrm{d}_\mu \operatorname{Im} f.$$

iv) f, g integrable, $\alpha, \beta \in \mathbb{C}, f \leq g$

$$\Rightarrow \int_X \mathrm{d}_\mu (\alpha f + \beta g) = \alpha \int_X \mathrm{d}_\mu f + \beta \int_X \mathrm{d}_\mu \text{ (Linearity)}$$

$$\Rightarrow \int_X \mathrm{d}_\mu f \leq \int_X \mathrm{d}_\mu g \text{ (Monotonicity)}$$

3.19. EXAMPLE. i) λ^d Lebesgue-Borel measure on $\mathbb{R}^d$, defined by

$$\lambda^d \left(\bigtimes_{\alpha=1}^{d}]a_\alpha, b_\alpha[\right) \coloneqq \prod_{\alpha=1}^{d} (b_\alpha - a_\alpha)$$

on rectangles and exteded to $\mathcal{B}^d \coloneqq \mathcal{B}(\mathbb{R}^d)$ by Theorem 6. If $f: \mathbb{R}^d \to \mathbb{C}$ Riemann-integrable $\Rightarrow f \lambda^d$-integrable and

$$\int_{\mathbb{R}^d} d^d x f(x) = \int_{\mathbb{R}^d} \lambda^d(\mathrm{d}x) f(x) \quad (\lambda^d(\mathrm{d}x) = \mathrm{d}^d x)$$

ii) Dirac measure on $\mathcal{B}^d$ on $\mathbb{R}^d$: Let $x_0 \in \mathbb{R}^d$ then

$$\delta_{x_0}(A) \coloneqq \begin{cases} 1, & x_0 \in A \\ 0, & x_0 \notin A \end{cases} \qquad \forall A \in \mathcal{B}^d$$

$\Rightarrow$ Every measurable $f: \mathbb{R}^d \to \overline{\mathbb{R}}$ is δ_{x_0}- integrable and

$$\int_{\mathbb{R}^d} \delta_{x_0}(\mathrm{d}x) f(x) = f(x_0).$$

3.20. LEMMA. *Consider $(X, \mathcal{A}, \mu)$ and $f: X \to \overline{\mathbb{R}}$ measurable, then the following are equivalent:*

i) *f integrable*
ii) *f_+, f_- integrable*
iii) *$|f|$ integrable*
iv) *$\exists u, v: X \to \overline{\mathbb{R}}$ integrable: $f_+(x) \leq u(x)$, $f_-(x) \leq v(x)$ for μ-almost all $x \in X$.*
v) *$\exists g: X \to \overline{\mathbb{R}}$ integrable: $|f(x)| \leq g(x)$ for μ-almost all $x \in X$*

3.21. THEOREM. *Consider $(X, \mathcal{A}, \mu)$ and $f: X \to \overline{\mathbb{R}}$ measurable*

i) *If f integrable and $\int d_\mu f < \infty$ then $\mu(\{x \in X: f(x) = \pm\infty\}) = 0$*
ii) *$\int d_\mu |f| = 0 \Leftrightarrow f(x) = 0$ for μ-almost all $x \in X$*

The benefit of general integration theory is the interchangeability of integration and limits.

3.22. THEOREM (Beppo Levi)(monotone convergence). *Let*

$$(f_n)_{n \in \mathbb{N}} \subset \overline{\mathcal{L}^1}(X, \mu) \coloneqq \{g: X \to \overline{\mathbb{R}}: g \text{ is } \mu\text{-integrable}\} \equiv \overline{\mathcal{L}^1}(X) \equiv \overline{\mathcal{L}^1}$$

with $0 \leq f_n \leq f_{n+1}$ $\forall n \in \mathbb{N}$ then

$$\lim_{n \to \infty} \int d_\mu f_n = \int d_\mu \lim_{n \to \infty} f_n$$

3.23. THEOREM (Fatou's Lemma). *Let* $(f_n)_n \subset \overline{\mathcal{L}^1}$ *with* $f_n \geq 0\ \forall n \in \mathbb{N}$*:*

$$\liminf_{n\to\infty} \int d_\mu f_n \geq \int d_\mu \liminf_{n\to\infty} f_n$$

3.24. THEOREM (Lebesgue)(dominated convergence). *Let* $(f_n)_n \subset \overline{\mathcal{L}^1}, f\colon X \to \overline{\mathbb{R}}$ *measurable,* $\lim_{n\to\infty} f_n(x) = f(x)$ *for* μ*-almost all* $x \in X$ *and* $\exists g \in \overline{\mathcal{L}^1} : |f_n(x)| \leq g(x),\ \forall x \in X \Rightarrow f \in \overline{\mathcal{L}^1}$ *and* $\lim_{n\to\infty} \int d_\mu f_n = \int d_\mu f$.

Useful notions of convergence:

3.25. DEFINITION. Let $f, f_n : X \to \overline{\mathbb{R}}$ measurable $\forall n \in \mathbb{N}$

i) Convergence (μ−) almost everywhere: $f_n \xrightarrow{\text{a. e.}} f$ (for μ−a. a. x) $\exists N \in \mathcal{A}$, $\mu(N) = 0$: $\lim_{n\to\infty} f_n(x) = f(x),\ \forall x \in X \setminus N$

ii) Stochastic Convergence (convergence in measure): $f_n \xrightarrow{\mu} f$, $\forall A \in \mathcal{A}$: $\mu(A) < \infty,\ \forall \epsilon > 0$

$$\lim_{n\to\infty} \mu\left(\{x \in A : |f_n(x) - f(x)| > \epsilon\}\right) = 0$$

iii) Convergence in L^1-norm: $f_n \xrightarrow{\|\cdot\|} f$, $\lim_{n\to\infty} \|f_n - f\|_1 = 0$ where

$$\|f\|_1 \coloneqq \int_x \mathrm{d}_x \|f\| \qquad L^1\text{-norm}$$

3.26. THEOREM.

$$\left.\begin{array}{r}\textit{convergence almost everywhere}\\ \textit{convergence in } L^1\textit{-norm}\end{array}\right\} \Rightarrow \textit{stochastic convergence}$$

A partial converse:

3.27. THEOREM. *Let* μ *be a* σ*-finite. Then* $f_n \xrightarrow{\mu} f \Rightarrow \forall$ *subsequences* $(f_{n_k})_k$ *of* $(f_n)_n \exists$ *a subsequence* $(f_{n_{k_l}})_l$ *such that* $f_{n_{k_l}} \xrightarrow{a.e.} f$ *as* $l \to \infty$

REMARK. Product spaces. Here only the product of two spaces. The generalisation to finitely many factors is obvious.

NOTATION. $(X_j, \mathcal{A}_j, \mu_j)$, $j = 1, 2$, *are measure spaces in the following.*

3.28. DEFINITION. Consider Cartesian product $X_1 \times X_2 \Rightarrow$ product σ-algebra

$$\mathcal{A}_1 \otimes \mathcal{A}_2 \coloneqq \sigma\left(\{A_1 \times A_2 :\ A_1 \in \mathcal{A}_1,\ A_2 \in \mathcal{A}_2\}\right) \subseteq \mathcal{P}\left(X_1 \times X_2\right)$$

3.29. THEOREM (Existence and uniqueness of σ-finite product measures). *Let* μ_1, μ_2, *be* σ*-finite, then* $\exists_1$ *measure on* $\mathcal{A}_1 \otimes \mathcal{A}_2$*, the product measure* $\mu_1 \otimes \mu_2$*:*

$$\mu_1 \otimes \mu_2(A_1 \times A_2) = \mu_1(A_1) \cdot \mu(A_2) \quad \forall A_j \in \mathcal{A}_j$$

3.30. EXAMPLE. For $\mathbb{R}^d = \mathbb{R} \times \cdots \times \mathbb{R}$ we have $\mathcal{B}(\mathbb{R}^d) = \mathcal{B}^d = \bigotimes_{i=1}^d \mathcal{B} = \mathcal{B}(\mathbb{R})$ and (Lebesgue-Borel measure on $\mathbb{R}^d$) $\lambda^d = \bigotimes_{i=1}^d \lambda$ (Lebesgue Borel-measure on $\mathbb{R}$)

3.31. THEOREM (Fubini-Tonelli). *Let* $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ *be* σ*-finite measure spaces and* $f\colon X_1 \times X_2 \to \overline{\mathbb{R}}$ *be* $\mathcal{A}_1 \otimes \mathcal{A}_2 - \overline{\mathcal{B}}$*-measurable, then:*
$x_1 \mapsto \int_{X_2} \mu_2(d_{x_2}) f(x_1, x_2)$ *is* $\mathcal{A}_1 - \overline{\mathcal{B}}$*-measurable and* $x_2 \mapsto \int_{X_1} (\mu_1(d_{x_1}) f(x_1, x_2)$ *is* $\mathcal{A}_2 - \overline{\mathcal{B}}$*-measurable. If one of the 3 integrals.*

$$\int_{X_1 \times X_2} \mu_1 \otimes \mu_2(dx_1 dx_2)\, |f(x_1, x_2)|$$

$$\int_{X_1} \mu_1(dx_1) \int_{X_2} \mu_2(dx_2)\, |f(x_1, x_2)|$$

$$\int_{X_2} \mu_2(dx_2) \int_{X_1} \mu_1(dx_1)\, |f(x_1, x_2)|$$

is finite, then so are the other two and

$$\int_{X_1\times X_2} \mu_1\otimes\mu_2(dx_1dx_2)\,|f(x_1,x_2)| = \int_{X_1}\mu_1(dx_1)\int_{X_2}\mu_2(dx_2)\,|f(x_1,x_2)|$$
$$= \int_{X_2}\mu_2(dx_2)\int_{X_1}\mu_1(dx_1)\,|f(x_1,x_2)|$$

3. L^p-Spaces

Notation ((this section)). *$(X,\mathcal{A},\mu)$ a measure space, $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, $f\colon X\to\mathbb{K}$ measurable.*

3.32. Definition (L^p-norm). i) For $p\in]0,\infty[$ set

$$\|f\|_p \coloneqq \left(\int \mathrm{d}\mu\,|f|^p\right)^{\frac{1}{p}} \text{ (possibly } \infty),$$

$$\|f\|_\infty \coloneqq \inf\{\alpha>0\colon \mu(\{x\in X\colon |f(x)|>\infty\})=0)\} = \inf_{\substack{\mathcal{N}\in\mathcal{A}\\ \mu(\mathcal{A})=0}}\ \sup_{x\in X\setminus\mathcal{N}}|f(x)|$$
$$= \operatorname*{ess\,sup}_{x\in X}|f(x)| \qquad \underline{\mu\text{-essential supremum}}$$

ii) For $p\in]0,\infty]$ let

$$\mathcal{L}^p=\mathcal{L}^p(\mu)=\mathcal{L}^p(X,\mu)=\mathcal{L}^p_{\mathbb{K}}\coloneqq\{f\colon X\to\mathbb{K}\colon f\text{ measurable and } \|f\|_p<\infty$$

We call these the μ-integrable functions.

iii) equivalence relation $\sim$ on $\mathcal{L}^p$: $f\sim g :\Leftrightarrow f=g$ a.e. $L^p\coloneqq\mathcal{L}^p/\sim$ "space of equivalence classes of p-integrable functions"

3.33. Theorem. *Let $f,g\colon X\to\mathbb{K}$ be measurable*

i) $\forall r,p,q\in[1,\infty]|\frac{1}{p}+\frac{1}{q}\colon \|fg\|_r\le\|f\|_p+\|g\|_q$ *Hölder's inequality*

ii) $\forall p\in[1,\infty]\colon \|f+g\|_p\le\|f\|_p+\|g\|_p$ *(Minkowski's inequality)*

iii) *If $\mu(X)<\infty$ and $1\le p\le q\le\infty$ then $\mathcal{L}^q\le\mathcal{L}^p$ and $\|f\|_p\le[\mu(x)]^{\frac{1}{p}-\frac{1}{q}}$*

Proof. i) **Case 1: $q=\infty$:** follows from $|fg|\le\|f\|_\infty\|g\|_\infty$

Case 2: $q<\infty$, $1\le p<\infty$: follows from $|fg|\le\|f\|_\infty\|g\|_\infty$

Without loss of generality assume $f,g\ge0$ and $\|f\|_p$, $\|g\|_q>0$

Claim: $\underbrace{x^{\frac{r}{p}}}_{u(x)}\le 1+\frac{r}{p}(x-1)=\underbrace{\frac{r}{p}x+\frac{r}{q}}_{v(x)}$ $\forall x\ge1,\ r\ge p$ because $u(1)=v(1)=1$ and $u'(x)=\frac{r}{p}x^{\frac{r}{p}-1}\le\frac{r}{p}=v'(x)$

Put $x=\frac{a}{b}$ in claim and $1\cdot\beta\Rightarrow \underline{\alpha^{\frac{r}{p}}\beta^{\frac{r}{q}}\le\frac{r}{p}\alpha+\frac{r}{q}\beta}$ $\forall\alpha\ge\beta>0$ (*)

- also true with $p\leftrightarrow q\Rightarrow$ condition $\alpha\ge\beta$ can be dropped.
- also true, if $\alpha=0$ or $\beta=0\Rightarrow$ (*) holds $\forall\alpha,\beta\ge0$.

apply (*) with $\alpha\coloneqq\frac{f^p}{\|f\|_p^p}$, $\beta\coloneqq\frac{g^p}{\|g\|_q^q}\Rightarrow\frac{(fg)^r}{\|f\|_p^r\|g\|_q^r}\le\frac{r}{p}\frac{f^p}{\|f\|_p^p}+\frac{r}{q}\frac{g^q}{\|g\|_q^q}$

$\xRightarrow{\int\mathrm{d}\mu}\frac{\|fg\|_r^r}{\|f\|_p^r\|g\|_q^r}\le\frac{r}{p}+\frac{r}{q}=1$

ii) Without loss of generality

- $f,g\ge0$ because $\|f+g\|_p\le\||f|+|g|\|_p$
- $\|f+g\|_p>0$ (otherwise nothing to prove)

Case 1: $p=\infty$: Note that for $\alpha,\beta>0$ and $\gamma=\alpha+\beta$

$$\mu(\{x\in X: f(x)>\alpha\})=\mu(\{f>\alpha\})=0 \text{ and } \mu(\{g>\beta\})=0$$

$$\Rightarrow \mu(\{f+g>r\})=0 \text{ Thus}$$

$$\begin{aligned}\|f+g\|_\infty &= \inf\{\gamma>0\colon \mu(\{f+g>\gamma\})=0\}\\ &\le \inf\{\alpha>0\colon \mu(\{f>\alpha\})=0\}+\inf\{\beta>0\colon \mu(\{g>\beta\})=0\}\\ &= \|f\|_\infty+\|g\|_\infty\end{aligned}$$

: $q \le p < \infty$

$$\begin{aligned}\|f+g\|_p^p &= \int \mathrm{d}\mu f(f+g)^{p-1} + \int \mathrm{d}\mu g(f+g)^{p-1}\\ &\overset{\text{Hölder}}{\le} \|f\|_p \left\|(f+g)^{p-1}\right\|_q + \|g\|_p \underbrace{\underbrace{\left\|(f+g)^{p-1}\right\|_q}_{\left(\int \mathrm{d}\mu (f+g)^{\overbrace{q(p-1)}^{p}}\right)^{\frac{p-1}{p}}}}_{\|f+g\|^{p-1}}\end{aligned}$$

iii) from i) with $g=1$ and $p \leftrightarrow r$: $\|f\|_p = \|f\|_q \|1\cdot f\|_p \le \overbrace{\|1\|_r}^{\mu(x)^{\frac{1}{p}-\frac{1}{q}}}$

□

3.34. Corollary. *If $\mu(x)<\infty$, then $\mathcal{L}^q$ is dense in $\mathcal{L}^p$,* $\forall 1\le p\le q\le\infty$

Proof. It suffices to prove $\mathcal{L}^\infty$ dense in $\mathcal{L}^p$, $\forall p\ge 1$ Let $f\in\mathcal{L}^p$ Without loss of generality $f\ge 0$ (otherwise decompose in $\operatorname{Re} f$, $\operatorname{Im} f$ in positive and negative parts) Let $f_n := min\{f,n\}\in\mathcal{L}^\infty$, $\forall n\in\mathbb{N}\Rightarrow$

$$\begin{aligned}\|f-f_n\|_p^p &= \int_X \mathrm{d}\mu\,(f-\min\{f,n\})^p \le \int_{\{f\ge n\}} \mathrm{d}\mu f^p\\ &= \int_X \mathrm{d}\mu f^p - \underbrace{\int_x \mathrm{d}\mu f^p \mathbb{1}_{\{f<n\}}}_{\xrightarrow{n\to\infty}\int_X \mathrm{d}\mu f^p} \xrightarrow{n\to\infty} 0\end{aligned}$$

Use dominated convergence or monotone convergence for the limit process in the last step. □

3.35. Theorem (Riesz-Fischer)). *Let $(X,\mathcal{A},\mu)$ be a measure space. Then L^p is a Banach space $\forall p\in[1,\infty]$ with norm $\|\cdot\|_p$. Moreover, L^2 is a Hilbert space with scalar product*

$$\langle f,g\rangle := \int_X d\mu(x)\overline{f(x)}g(x) \quad \forall f,g\in L^p$$

3.36. Warning. Notation does not distinguish between equivalence classes and representation.

Proof. $\forall p\in[1,\infty]$, L^p is a normed space:

- $\|f\|_p = 0 \Rightarrow f(x)=0$ for a.e. $x\in X$, i.e. $f=0$ in L^p
- $\|\alpha f\|_p = |\alpha|\,\|f\|_p$ clear
- triangle inequality from Thm 33 ii) ($p\ge 1!$)

and $\langle\cdot,\cdot\rangle$ si a scalar product on L^2 with $\langle f,f\rangle = \|f\|_2^2$ All that remains to be shown is the completeness $\forall p\in[1,\infty]$

Case 1: $p=\infty$: Let $(f_n)_n\subset L^\infty$ be Cauchy,

$\forall n,m\in\mathbb{N}\ \exists N_{n,m}\in\mathcal{A},\ \mu(N_{n,m})=0\colon |f_n(x)-f_m(x)|\le \|f_n-f_m\|_\alpha,\ \forall x\in N_{n,m}^c$

$\Rightarrow N := \bigcup_{n,m\in\mathbb{N}} N_{n,m}$ is a null set and $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{K}$, uniformly$\forall x\in N^c(*)$ $\Rightarrow$ (completeness of $\mathbb{K}$) $X\ni x\mapsto f(x) := \begin{cases}\lim_{n\to\infty} f_n(x), & x\in N^c\\ 0, & x\in N\end{cases}$

well defined measurable and

$$\begin{aligned}\|f - f_n\|_\infty &\le \sup_{x\in N^c} |f(x) - f_n(x)| \\ &= \sup_{x\in N^c} \lim_{m\to\infty} |f_m(x) - f_n(x)| \\ &\le \sup_{x\in N^c} \sup_{m\ge n} |f_m(x) - f_n(x)| \\ &\le \sup_{x\in N^c} \sup_{y\in N^c} |f_m(y) - f_n(y)|\end{aligned}$$

$\overset{(*)}{\Rightarrow} \lim_{m\to\infty} \|f - f_n\|_\infty = 0$

Case 2: $1 \le p < \infty$: : Let $(f_n)n \subset L^p$ be Cauchy. $\exists (n_k)_k \subset N$ such that: $\|f_{n_k} - f_{n_{k+1}}\|_p < 2^{-k}\ \forall k \in \mathbb{N}$. Now: $h \coloneqq |f_{n_1}| + \sum_{k=1}^\infty |f_{n_{k+1}} - f_{n_k}| \in L^p$ because $\|h\|_p \underbrace{\le}_{\text{triangle}} \|f_{n_{k+1}}\|_p + \underbrace{\sum_{k=1}^\infty 2^{-k}}_{1} < \infty$.

In particular $\sum_{k=1}^\infty |f_{n_{k+1}} - f_{n_k}| \in L^p$

$\overset{\text{Thm. 20}}{\Longrightarrow} \sum_{k=1}^\infty |f_{n_{k+1}}(x) - f_{n_k}(x)| < \infty$ for μ-a.a. $x \in X$

$\Rightarrow (f_{n_k}(x))_k \subset \mathbb{K}$ is Cauchy for μ-a.a. $x \in X$, because

$$\left|f_{n_{k_2}} - f_{n_{k_1}}(x)\right| \le \sum_{j=k_1}^{k_2-1} \left|f_{n_{j+1}}(x) - f_{n_j}(x)\right| \le \sum_{j=k_1}^{\infty} |\ldots| \xrightarrow{k_1\to\infty} 0$$

$\overset{\mathbb{K}\text{ complete}}{\Longrightarrow} \exists f : X \to \mathbb{K}$ measurable with $f(x) = \lim_{k\to\infty} f_{n_k}(x)$ for μ-a.a. $x \in X$. Moreover, $|f_{n_k}| \le h \quad \forall k \in \mathbb{N}$ since $f_{n_k} = f_{n_1} + \sum_{j=1}^{k-1} \left(f_{n_{j+1}} - f_{n_j}\right)$ $\Rightarrow |f| \le h$ a.e. $\Rightarrow |f_{n_k}|^p, |f|^p \in L^1$

Let $g_k \coloneqq |f_{n_k} - f|^p \Rightarrow g_k \xrightarrow{k\to\infty} 0$ a.e. and $0 \le g_k \le (|f_{n_k}| + |f|)^p \le 2^p |h| \in L^1$, $\forall k \in \mathbb{N} \Rightarrow$ dominated convergence $\lim_{k\to\infty} \int_X \mathrm{d}\mu g_k = 0 \Rightarrow$ subsequence $(f_{n_k})_k$ converges to f in $\|\cdot\|_p$

$(f_n)_n$ Cauchy $\Rightarrow (f_n)_n$ converges to f in $\|\cdot\|_p$

□

Some preparations to indentify the dual space of L^p:

3.37. Theorem (Clarkson inequalities). *Let $p \in]1, 2]$, $f, g \in L^P$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left\|\frac{f+g}{2}\right\|_p^q + \left\|\frac{f-g}{2}\right\|_p^q \le \left(\frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p\right)^{q-1} \qquad (1)_<$$

$$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^q \ge \left(\frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p\right)^{q-1} \qquad (2)_<$$

For $p \in [2, \infty[$, the inequalities are reversed! $\to (1)_>, (2)_>$

Proof. See e.g. R. A. Adams, Sobolev spaces, Thm. 2.28. See also Hirzebruch/Scharlau §17 or Hanno's inequality in Lieb/Loss Analysis, Sec. 2 ($p = 2 :$ parallelogram identity) Note: The Clarkson's inequalities in Adams are stated for Lebesgue measure, but the proof is valid for general measure spaces $(X, \mathcal{A}, \mu)$. □

3.38. Theorem. *Let $p \in]1, \infty[$. Then L^p is uniformly convex, that is $\forall \epsilon > 0$, $\exists \delta > 0$, $\forall f, g \in L^p$ with $\|f\|_p = \|g\|_p = 1$ and $\|f - g\|_p \ge \epsilon$: $\left\|\frac{f+g}{2}\right\|_p \le 1 - \delta$*

Proof. **Case $p \le 2$:** : $1)_< \Rightarrow \left\|\frac{f+g}{2}\right\|_p^q \le 1 - 2^{-q}\epsilon^q$

Case $p \ge 2$: : $2)_> \Rightarrow \left\|\frac{f+g}{2}\right\|_p^p \le 1 - 2^{-p}\epsilon^p$

□

3.39. THEOREM (Riesz representation for L^p). *Let $p \in [1,\infty[$, $(X,\mathcal{A},\mu)$ a measure. If $p = 14$, assume μ is σ-finite. Then the mapping* $(\frac{1}{p} + \frac{1}{q} = 1)$ *$J : L^q \to (L^p)^*$, $f \mapsto l_f$ where $l_f(g) := \int d\mu f g$, $\quad g \in L^p$ is an isometric isomorphism $(L^p)^* \simeq L^q$.*

PROOF. J is linear (clear). Now let $f \in L^q$.
Well defined: l_f linear, Hölder $\Rightarrow |l_f(g)| \leq \|f\|_q \|g\|_p \Rightarrow \|l_f\|_{(L^p)^*} \leq \|f\|_q \Rightarrow l_f \in (L^p)^*$
Claim: $\|l_f\|_{(L^p)^*} = \|f\|_q$; (hence J injective)

Case $p > 1$: Let $\tilde{g} := \overline{f}|f|^{q-2}$ $(\leftarrow := 0$ where $f = 0) \Rightarrow |\tilde{g}|^p = |f|^{p(q-1)} = |f|^q \in L^1 \Rightarrow \tilde{g} \in L^p$ and $\|\tilde{g}\|_p = \|f\|_q^{\frac{q}{p}} \Rightarrow \frac{l_q(\tilde{g}}{\|\tilde{g}\|} = \frac{\|f\|_q^q}{\|f\|_q^{\frac{q}{p}}} = \|f\|_q \Rightarrow$ claim.

Case $p = 1$: μ is σ-finite $\Rightarrow X = \bigcup_{n\in\mathbb{N}} X_n$, $X_n \subset X_{n+1}$, $\forall n$ and $\mu(X_n) < \infty$. Let $0 < \epsilon < \|f\|_\infty \Rightarrow M := \{x \in X : |f(x)| \geq \|f\|_\infty - \epsilon\}$. M is measurable by definition. $\mu(M) > 0$ and $\exists N \in \mathbb{N} : \mu(M \cap X_N) \in]0,\infty[$. Let $\tilde{g} := \frac{\overline{f}}{|f|}\frac{1}{\mu(M_N)}\int_{M_N} \mathrm{d}\mu |f| \geq \|f\|_\infty - \epsilon \Rightarrow \|l_f\|_{(L^1)^*} \geq \|f\|_\infty - \epsilon$, $\forall \epsilon > 0 \Rightarrow$ claim.
Remains to be proven: J surjective! Let $\xi \in (L^p)^*$ with $\|\xi\|_{(L^p)^*}$ Without loss of generality

Case $p > 1$: $\exists (f_k)_k \subset L^p$, $\|f_k\|_p = 1 \forall k : |\xi(f_n)| \xrightarrow{k\to\infty} 1$ Without loss of generality assume $\xi(f_k) \geq 0$ (multiply f_k by appropriate phase factor!)
<u>Claim:</u> $(f_k)_k$ is Cauchy!
for, assume $\exists \epsilon > 0$ and subsequence (k_j), $(k_j')_j$ such that:

$$\left\|f_{k_j} - f_{k_j'}\right\|_p \geq \epsilon \quad \forall j$$

uniform convexity (Thm. 36) $\Rightarrow \exists \delta > 0$ such that $\left\|\frac{f_{k_j}+f_{k_j'}}{2}\right\|_p \leq 1 - \delta$

$$\begin{aligned}\Rightarrow 1 \geq & \xi\left(\frac{f_{k_j} + f_{k_j'}}{\left\|f_{k_j} + f_{k_j'}\right\|_p}\right) = \frac{1}{\left\|f_{k_j} + f_{k_j'}\right\|_p}\left(\xi(f_{k_j}) + \xi(f_{k_j'})\right) \\ & \geq \frac{1}{1-\delta}\frac{1}{2}\left(\xi(f_{k_j}) + \xi(f_{k_j'})\right) \xrightarrow{j\to\infty} \frac{1}{1-\delta} > 1 ↯\end{aligned}$$

Thus $\exists f \in L^p$: $f_k \xrightarrow{n\to\infty} f$ in L^p and $\|f\|_p = 1$ (N. B. $\xi(f) = 1$) Consider l_g for $g := |f|^{p-2}\overline{f}$ (0, where f is 0)

- $\|g\|_q = \|f\|_p = 1 \overset{\text{above}}{\Longrightarrow} \|l_g\|_{(L^p)^*} = 1$
- $l_g(f) = 1$

Now $\xi = l_g$ follows from
<u>Claim</u> Let ξ_1, $\xi_2 \in (L^p)^*$ with $\|\xi_1\|_{(L^p)^*} = 1$. Suppose $\exists f \in L^p$, $\|f\|_p = 1$ with $\xi_1(f) = \xi_2(f) = 1$. $\Rightarrow \xi_1 = \xi_2$. For suppose not, then $\exists \tilde{h} \in L^p$ With $\xi_1(h) = 1$, $\xi_2(h) = -1$. because $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}\xi_1(\tilde{h})\\\xi_2(\tilde{h})\end{pmatrix}$ are linearly independent in $\mathbb{K}^2$. There fore $\xi_1(f+th) = \xi_2(f-th) = 1+t, \forall t \geq 0, \overset{\|\xi_j\|=1}{\Rightarrow} \|f \pm th\|_p \geq 1+t$
Clarkson (Thm. 35) $\Rightarrow \forall t \geq 0$

$p \in]1,2]$:

$$\overbrace{1+t^p}^{\alpha(t)} \|h\|_p^p = \left\|\frac{(f+th)+(f-th)}{2}\right\|_p^p + \left\|\frac{(f+th)-(f-th)}{2}\right\|_p^p$$
$$\overset{(2)<}{\geq} \frac{1}{2}\|f+th\|_p^p + \frac{1}{2}\|f-th\|_p^p \geq \underbrace{(1+t)^p}_{\beta(t)}$$

↯because $\alpha(0) = \beta(0) = 1$ and $pt^{p-1}\|h\|_p^p = \alpha'(t) < \beta'(t) = p(1+t)^{p-1}$ $\forall t$ sufficiently small.

$p \in [2,\infty[$:

$$1+t^1\|h\|_p^q = \left\|\frac{(f+th)+(f-th)}{2}\right\|_p^q + \left\|\frac{(f+th)-(f-th)}{2}\right\|_p^q$$
$$\overset{(1)>}{\geq} \left(\frac{1}{2}\|f+th\|_p^p + \frac{1}{2}\|f-th\|_p^p\right)^{q-1} \geq (1+t)^q$$

↯as above. $\Rightarrow$ Claim is true $\Rightarrow J$ surjective for $p > 1$

Case $p = 1$: (1) Assume first $\mu(X) < \infty$. Let $\tilde{p} > 1$ and $f \in L^\infty \overset{\text{Thm. 32 iii)}}{\subset} L^P \overset{\text{Thm. 32 iii)}}{\subset} L^1$

$\Rightarrow |\xi(f)| \leq \underbrace{\|\xi\|_{(L^1)^*}}_{1} \|f\|_1 \leq \underbrace{[\mu(X)]^{1-\frac{1}{\tilde{p}}}}_{=:M_{\tilde{p}}} \|f\|_{\tilde{p}}$

$\Rightarrow \xi \in (L^{\tilde{p}})^* \overset{\text{above}}{\Rightarrow} \exists_1 g_{\tilde{p}} \in L^{\tilde{q}}$ such that $\xi|_{L^{\tilde{p}}} = l_{g_{\tilde{p}}}$ and $\|g_{\tilde{p}}\|_{\tilde{q}} \leq M_{\tilde{p}}$. Thus $\forall p_1, p_2 > 1,\ \forall f \in L^\infty$

$$l_{g_{p_1}}(f) = l_{g_{p_2}}(f) \text{ i.e. } \int_x \mathrm{d}\mu(g_{p_1} - g_{p_2})f = 0$$

Choose $f = \operatorname{sgn}(g_{p_1} - g_{p_2}) \Rightarrow g_{p_1} = g_{p_2} = g$ independent of $\tilde{p}$! and $\|g\|_{\tilde{q}} \leq M_{\tilde{p}} = \underbrace{\mu(X)^{\frac{1}{\tilde{q}}}}_{\text{bounded in } \tilde{q}}$ $\forall \tilde{q} \in]1,\infty[\overset{\text{Exercise}}{\Rightarrow} g \in L^\infty$ and $\|g\|_\infty \leq \lim_{\tilde{q}\to\infty} \mu(x)^{\frac{1}{\tilde{q}}} = 1 \Rightarrow l_g \in (L^1)^*$ Corollar 33 $\Rightarrow L^{\tilde{p}}$ dense in L^∞, i.e. ξ and l_g agree on a dense subspace $\overset{\text{Thm. 2.30}}{\Rightarrow} \xi = l_g$ on L^1

(2) relax finiteness assumption: assume μ is σ-finite $X = \bigcup_{n\in\mathbb{N}} X_n$, $\mu(X_n) < \infty, \forall n \in \mathbb{N}$. Let $\xi \in L^1(X)^* \Rightarrow \xi \in L^1(X_n)^*, \forall n \in \mathbb{N} \Rightarrow \exists_1 g_n \in L^\infty(X_n)$ such that $\xi = l_{g_n}$ on $L^1(X_n)$ and $\|\xi\|_{L^1(X_n)^*} \geq \|g_n\|_{L^\infty(X_n)}$, $\forall n \in \mathbb{N}$ Define $g \coloneqq \sum_{n\in\mathbb{N}} \tilde{g}_n$, $\tilde{g}_n(x) \coloneqq \begin{cases} g_n(x) & x \in X_n \\ 0 & x \notin X_n \end{cases} \Rightarrow \|g\|_{L^\infty(X)} \leq \|\xi\|_{L^1(X)^*}$ For $f \in L^1(X)$ let $f_N \coloneqq \sum_{n=1}^N f\mathbb{1}_{X_n} \Rightarrow \|f - f_n\|_1 = \int_X \mathrm{d}\mu |f| \sum_{n=N+1}^\infty \mathbb{1}_{X_n} \xrightarrow[\text{dom. cvg.}]{N\to\infty} 0 \Rightarrow \xi(f) = \lim_{n\to\infty} \xi(f_N) = \sum_{n\in\mathbb{N}} \underbrace{\xi(f\mathbb{1}_{X_n})}_{l_{g_n}(f)} = \sum_{n\in\mathbb{N}} \underbrace{\int_{X_n} \mathrm{d}\mu g_n f}_{\int_X \mathrm{d}\mu \tilde{g}_n f} \left|\sum_{n=1}^N \tilde{g}_n f\right| \leq |gf| \in L^1 \overset{\text{dom. cfg.}}{=} \int_X \mathrm{d}\mu g f = l_g(f)$

□

Next we turn to separability of L^p-spaces.

3.40. Theorem. *Let X be a compact metric space and μ a finite Borel measure on X. THen $L^p(X,\mu)$ is separable $\forall p \in [1,\infty[$ (wrong for $p = \infty$, execpt when μ is "trivial")*

(See Cor. 41 for a generalisation)
An important concept in the proof: (p.t.o.)

IDEA. Use Lusin's theorem (see e.g. Thm. 2.33 in W. Rudin, Real and complex analysis) to show that $\mathcal{C}(X)$ is dense in L^p with respect to $\|\cdot\|_p$. (Requires regularity of μ! But this follows from Thm. 2.18 in Rudi, as every open subset A of a compact metric space is σ-compact: $A = \bigcup_n \{x \in A : \text{dist}(\delta A, x) \geq \frac{1}{n}\}$). But $\mathcal{C}(X)$ is separable by Thm. 1.48. □

3.41. DEFINITION. Let (X,μ) be a topological Hausdorff space, $\mathcal{B} = \sigma(\mathcal{T})$ the Borel σ-algebra and μ a Borel measure on X

- $\boldsymbol{\mu}$ **is inner regular:** $:\Leftrightarrow \forall B \in \mathcal{B},\ \mu(B) = \sup_{\substack{K \subset B \\ K \text{ compact}}} \mu(K)$.
- $\boldsymbol{\mu}$ **is outer regular:** $:\Leftrightarrow \forall B \in \mathcal{B},\ \mu(B) = \inf_{\substack{U \supset B \\ U \text{ open}}} \mu(U)$.
- $\boldsymbol{\mu}$ **is regular:** $:\Leftrightarrow \mu$ is inner regular and outer regular.

3.42. EXAMPLE. Every Borel measure on $\mathbb{R}^d$ is regular. (see H. Bauer, Maß- u. Integrationstheorie, p. 176)

3.43. COROLLARY. *Let X be a metric space and μ a Borel measure on X. Assume $X = \bigcup_{n\in\mathbb{N}} X_n$ where X_n compact and $X_n \subset X_{n+1}$, $\forall n \in \mathbb{N}$ ($\Rightarrow \mu$ is σ-finite!). Then $L^p(X,\mu)$ is separable $\forall p \in [1,\infty[$.*

PROOF. Exercise (Given Thm. 38) □

Thm. 38 relies on density of $\mathcal{C}(X)$ in $L^p(X)$. For $X = \Omega \subseteq \mathbb{R}^d$ open and μ Lebesgue-Borel, even arbitrarily often differentiable functions will be dense in L^p ... This needs preparations:

3.44. THEOREM (Young's inequality). *Let $p,q,r \in [1,\infty]$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$ (*)
Then $\forall f \in L^p(\mathbb{R}^d),\ g \in L^q(\mathbb{R}^d),\ h \in L^r(\mathbb{R}^d)$*

$$\begin{aligned} I &:= \left| \int_{\mathbb{R}^d\times\mathbb{R}^d} dx\,dy\, f(x) g(x-y) h(y) \right| \\ &\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} dx\,dy\, |f(x)|\,|g(x-y)|\,|h(y)| \leq \|f\|_p \|g\|_q \|h\|_r \end{aligned}$$

PROOF. A smart application of Hölder ... Without loss of generality assume $f,g,h \geq 0$. Let p',q',r' be conjugate to p,q,r

$$\overset{(*)}{\Rightarrow} \underbrace{\frac{1}{p} + \frac{1}{p'}}_{1} + \underbrace{\frac{1}{q} + \frac{1}{q'}}_{1} + \underbrace{\frac{1}{r}}_{1-\frac{1}{r'}} = 2 + \frac{1}{p'} + \frac{1}{q'} \Rightarrow \frac{1}{p'} + \frac{1}{q'} + \frac{1}{r'} = 1$$

Case: $\boldsymbol{p,q,r \in\,]1,\infty[}$:

$$\left.\begin{aligned} \alpha(x,y) &:= f(x)^{\frac{p}{r'}} g(x-y)^{\frac{q}{r'}} \\ \beta(x,y) &:= g(x-y)^{\frac{q}{p'}} h(y)^{\frac{r}{p'}} \\ \gamma(x,y) &:= f(x)^{\frac{p}{q'}} h(y)^{\frac{r}{q'}} \end{aligned}\right\} \Rightarrow f(x)g(x-y)h(y) = \alpha\beta\gamma$$

Hölder in L^1 $(\mathbb{R}^d \times \mathbb{R}^d,\ \mathrm{d}x\mathrm{d}y)$

$$I = |\alpha\beta\gamma|_1 \leq \|\alpha\|_{r'} \|\beta\gamma\|_r \leq \|\alpha\|_{r'} \|\beta\|_{p'} \|\gamma\|_{q'} \quad (1)$$

$$\text{but } \|\alpha\|_{r'}^{r'} = \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}y f(x)^p g(x-y)^q$$
$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}^d} \mathrm{d}x f(x)^p \underbrace{\int_{\mathbb{R}^d} \mathrm{d}y g(x-y)^q}_{\int_{\mathbb{R}^d} \mathrm{d}z g(z)^q} = \|f\|_p^p \|g\|_q^q$$

$$\text{similarly } \|\beta\|_{p'} = \|g\|_q^{\frac{q}{p'}} \|h\|^{\frac{r}{q'}}$$
$$\|\gamma\|_{q'} = \|f\|_p^{\frac{p}{q'}} \|h\|_r^{\frac{r}{q'}}$$

in (1) $\Rightarrow$ Claim.

Case: $p = \infty$: $\Rightarrow q = r = 1$ (similarly: $q = \infty$, $r = \infty$) $\Rightarrow$ Claim clear (Take out $\|f\|_\infty$)

Case: $p = 1 \Rightarrow \frac{1}{q} + \frac{1}{r} = 1$: (Similarly: $q = 1$ or $r = 1$) apply Hölder to $\int \mathrm{d}y g(x-y)h(y)$

□

3.45. Theorem. *Let $1 \le \frac{1}{p} + \frac{1}{q} \le 2$, $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$. Then $(f * g)(x) := \int_{\mathbb{R}^d} \mathrm{d}y f(y)g(x-y)$ exists for Lebesgue-a.a. $x \in \mathbb{R}^d$ and defines an element (equivalence class) $f * g \in L^{r'}(\mathbb{R}^d)$, the convolution of f and g. Here, $\frac{1}{r'} = -1 + \frac{1}{p} + \frac{1}{q}$ (notation as in Thm. 42). Moreover $f * g = g * f$ and $\|f * g\|_{r'} \le \|f\|_p \|g\|_q$.*

Proof. Let $h \in L^r(\mathbb{R}^d) \overset{\text{Young}}{\Rightarrow} \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, |f(y)|\,|g(x-y)|\,|h(x)|\, \|f\|_p \|g\|_q \|h\|_r < \infty \overset{\text{Fubini}}{\Rightarrow} x \mapsto |h(x)| \int_{\mathbb{R}^d} \mathrm{d}y\, |g(x-y)|\,|f(y)| \in L^1(\mathbb{R}^d)(*) \Rightarrow (f * g)(x)$ exists for Lebesgue-a.a. $x \in \mathbb{R}^d$

Case $r = \infty$: Set $h(x) = 1 \Rightarrow$ dense

Case $r < \infty$: $\overset{\text{Fubini}}{\Rightarrow} l(h) := \int_{\mathbb{R}^d} (f * g)(x)h(x)$

- exists
- is linear in h
- $|l(h)| \le \|f\|_p \|g\|_q \|h\|_r$

$\Rightarrow l \in L^r(\mathbb{R}^d)^* \overset{\text{Thm. 37}}{\Rightarrow} \exists_1 \xi \in L^{r'}(\mathbb{R}^d)$ such that $l(h) = \int_{\mathbb{R}^d} \mathrm{d}x \xi(x)h(x)$, $\forall h \in L^r(\mathbb{R}^d) \Rightarrow \xi = f * g \in L^{r'}(\mathbb{R}^d)$ and $\|f * g\|_{r'} = \|l\|_{l^{r*}} \le \|f\|_p \|g\|_q$

Commutativity:

$$(f * g)(x) = \int_{\mathbb{R}^d} \mathrm{d}y f(y)g(x-y) \overset{z=x-y}{=} \int_{\mathbb{R}^d} \mathrm{d}z f(x-z)g(z)$$
$$= (g * f)(x)$$

□

3.46. Definition. Let $\varnothing \ne \Omega \subseteq \mathbb{R}^d$ be open and $k \in \mathbb{N}$.

$$\mathcal{C}^k(\Omega) := \{f \in \mathcal{C})\Omega) : \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d} f \in \mathcal{C}(\Omega) \forall \alpha_1, \cdots \alpha_d \in \mathbb{N}_0 \text{ with } \sum_{j=1}^d \alpha_j \le k\}$$

k-times continuously differentiable functions.

$$\mathcal{C}^\infty(\Omega) := \bigcap_{k\in\mathbb{N}} \mathcal{C}^k(\Omega) \text{ arbitrarily often differntiable functions}$$

for $k \in \mathbb{N} \cup \{\infty\}$:

$$\mathcal{C}_0^k(\Omega) := \{f \in \mathcal{C}^k(\Omega) : \forall \epsilon > 0,\ \exists K_{\epsilon,f} \subset \Omega \text{ compact such that} |f(x)| < \epsilon,\ \forall x \notin K_{\epsilon,f}\}$$

where the 0 indicates that the functions converge to 0 towards the boundary.

$$\mathcal{C}_C^k(\Omega) := \{f \in \mathcal{C}^k(\Omega) : \exists K_f \subset \Omega \text{ compact such that} f(x) = 0,\ \forall x \notin K_f\}$$

Here C indicates compact support, i.e. the functions vanish in a "neighbourhood" of the boundary.

$$\underline{\text{support of f:}}\quad \operatorname{supp}(f) := \overline{\{x \in \Omega\colon f(x) \neq 0\}}\ \text{(closed!)}$$

The main result:

3.47. THEOREM. *Let* $\varnothing = \Omega \subseteq \mathbb{R}^d$ *be open and* $p \in [1,\infty[$. *Then* $\mathcal{C}_C^\infty(\Omega)$ *is dense in* $L^p(\Omega,\lambda^d)$, *with respect to* $\|\cdot\|_p$

3.48. REMARK.
- $L^p(\Omega)$ is the completion of $\mathcal{C}_C^\infty(\Omega)$ with respect to $\|\cdot\|_p$, $(p<\infty)$
- $L^P(\Omega)$ is separable, because $\mathcal{C}_C^\infty(\Omega)$ is so.
- The completion of $\mathcal{C}_C^\infty(\Omega)$ with respect to $\|\cdot\|_\infty$ is $\mathcal{C}_0^\infty(\Omega)$.

The proof of Thm. 45 relies on mollifier techniques

3.49. THEOREM. *Let* $p \in [1,\infty]$, $f \in L^p(\Omega)$ *(*Ω *open in* $\mathbb{R}^d$*), with compact support in* Ω, *let* $j \in \mathcal{C}_C^\infty(\mathbb{R}^d)$, $j \geq 0$, $\int_{\mathbb{R}^d} dx j(x) = 1$. *For* $\epsilon > 0$ *let* $j_\epsilon(x) := \epsilon^{-d} j(\frac{x}{\epsilon})$. *Then* $\forall \epsilon > 0$ *sufficiently small:* $f_\epsilon := f * j_\epsilon \in \mathcal{C}_C^\infty(\Omega)$ *and* $\|f_\epsilon\|_p \leq \|f\|_p$.

PROOF. Extend f to $\mathcal{C}_C^\infty(\mathbb{R}^d)$ by setting it to zero outside its support $\Rightarrow^\dagger$ $f_\epsilon \in L^p(\mathbb{R}^d)$ with $\|f_\epsilon\|_p \leq \|f\|_p \underbrace{\|j_\epsilon\|_1}_{\|j\|_1=1}$ $f_\epsilon(x) = \int_{\mathbb{R}^d} \mathrm{d}y j_\epsilon(x-y) f(y)$ let $\delta := \operatorname{dist}(\operatorname{supp} f, \delta\Omega)$ (∞allowed) $\exists \epsilon_0 > 0,\ \forall \epsilon \in]0,\epsilon_0] : \operatorname{supp}(j_\epsilon) \subset B_{\frac{\delta}{2}(0)} \Rightarrow \operatorname{supp}(f_\epsilon)$ compact in Ω derivatives: let $0 \neq z \in \mathbb{R}^d$. Since $j_\epsilon \in \mathcal{C}_C^\infty(\mathbb{R}^d) \Rightarrow$

- $\frac{1}{|z|}(f_\epsilon(x+z) - f_\epsilon(x)) = \int_{\mathbb{R}^d} \mathrm{d}y \underbrace{\frac{j_\epsilon(x+z-y) - j_\epsilon(x-y)}{|z|}}_{|\cdot| \leq \gamma \mathbb{1}_K(y);\gamma,K \text{ independent of } y,z} f(y)$
- limit $|z| \to 0$ exists inside $\int \overset{\text{dom. cvg.}}{\Rightarrow}$ can be exchanged with $\int \Rightarrow 1^{\text{st}}$ partial derivatives of f_ϵ exist.

$\Rightarrow f_\epsilon \in \mathcal{C}_C^\infty(\Omega)$ follows from induction. □

3.50. LEMMA. *Let* $f \in L^p(\Omega)$, $\operatorname{supp}(f)$ *compact in* Ω *and* f_ϵ *as in Lemma 47. Then* $\lim_{\epsilon>0} \|f - f_\epsilon\|_p = 0, \quad \forall p \in [1,\infty[$.

PROOF. Let $f \in L^p(\Omega)$, $K := \operatorname{supp}(f)$ compact in Ω. Let $\eta > 0$. Without loss of generality assume $f \geq 0$ (otherwise decompose)

Step 1: cut off at large values:

$$\exists M > 0 \text{ such that } \left\|f - f^{(M)}\right\|_p \leq \frac{\eta}{3} \text{ where } f^{(M)} := \min(f, M)$$

by dominated convergence: $0 \leq f - f^{(M)} \leq f$, $\lim_{M\to\infty}\left(f(x) - f^{(M)}(x)\right) = 0$ a.e. $x \in S_C$

Step 2: Approximate by a simple functions $\exists s_N = \sum_{n=1}^N \underbrace{\alpha_n}_{\leq M} \mathbb{1}_{\underbrace{A_n}_{\subseteq K}}$ with $\left\|f^{(M)} - s_N\right\|_p \leq \frac{\eta}{3}$. By the definition of the integral: $\left\|f^{(M)} - s_N\right\|_1 \xrightarrow{N\to\infty} 0$ for sequence of simple functions $s_N \leq f^{(M)}$ and $\left\|f^{(M)} - s_N\right\|_p^p \leq \left\|f^{(M)} - s_N\right\|_1 \underbrace{\left\|f^{(M)} - s_N\right\|_\infty^{p-1}}_{\leq M^{p-1}}$

†Theorem 42 and with $r' = p$ and $q = 1$

Step 3: approximate the indicator function by $\mathcal{C}_C^\infty(\Omega)$

$$\exists\epsilon>0\colon \forall n\in\{1,\dots,N\}\qquad \Bigg\|\mathbb{1}_{A_n}-\underbrace{j_\epsilon*\mathbb{1}_{A_n}}_{\in\mathcal{C}_C^\infty(\Omega)\text{ by Lemma 47)}}\Bigg\|_p\le\frac{\eta}{3MN}$$

Steps 1,2,3 give the claim because with

$$f_\epsilon:=\sum_{n=1}^N\alpha_n j_\epsilon*\mathbb{1}_{A_n}\in\mathcal{C}_C^\infty(\Omega)$$

we have

$$\|f-f_\epsilon\|_p\le\underbrace{\left\|f-f^{(M)}\right\|_p}_{<\frac{\eta}{3}}+\underbrace{\left\|f^{(M)}-S_N\right\|_p}_{<\frac{\eta}{3}}+\underbrace{\|s_N-f_\epsilon\|_p}_{\le\sum_{n=1}^N\underbrace{\alpha_n}_{\le M}\underbrace{\|\mathbb{1}_{A_n}-j_\epsilon*\mathbb{1}_{A_n}\|_p}_{\le\frac{\eta}{3MN}}<\frac{\eta}{3}}<\eta$$

□

The case of the argument uses regularity of Lebesgue-Borel measure λ^d

3.51. Lemma. *In the situation of Lemma 47 we have $\forall p\in[1,\infty[;\ \forall A\in\mathcal{B}(\mathbb{R}^d)$ bounded*

$$\lim_{\epsilon\searrow 0}\|\mathbb{1}_A-j_\epsilon*\mathbb{1}_A\|_p=0$$

Proof. **1^{st} act:** it suffices to prove the Lemma for $p=2$ because if $g\in L^\infty$ with $\operatorname{supp}(g)$ bounded ($\Rightarrow g\in L^p$), then

$p\ge 2$: $\|g\|_p^p\le\|g\|_\infty^{p-2}\|g\|_2^2$

$1\le p\le 2$: $\|g\|_p^p=\int\mathrm{d}x\,|g|^p\,\mathbb{1}_{\operatorname{supp}(g)}\le^\dagger\|g\|_2^p\,\lambda^d(\operatorname{supp}(g))^{1-\frac{p}{2}}$

2^{nd} act:

$$\int_{\mathbb{R}^d}\mathrm{d}x j_\epsilon(x)=\int_{\mathbb{R}^d}\mathrm{d}x\epsilon^{-d}j(\frac{x}{\epsilon})=^\ddagger\int_{\mathbb{R}^d}\mathrm{d}y j(y)=1$$

$$\Rightarrow|\mathbb{1}_A(x)-(j_\epsilon(x)*\mathbb{1}_A)(x)|=\left|\int_{\mathbb{R}^d}\mathrm{d}y\,\underbrace{j_\epsilon(y)}_{\sqrt{\cdot}\sqrt{\cdot}}\underbrace{[\mathbb{1}_A(x)-\mathbb{1}_A(x-y)]}_{=:G(x,y)}\right|$$

$$\overset{\text{CSI}}{=}\underbrace{\|j_\epsilon\|_1^{\frac12}}_{1}\left(\int_{\mathbb{R}^d}\mathrm{d}y j_\epsilon(y)G(x,y)^2)\right)^{\frac12}$$

$$\Rightarrow\|\mathbb{1}_A-j_\epsilon*\mathbb{1}_A\|_2^2\le\int_{\mathbb{R}^d}\mathrm{d}y j_\epsilon(y)\int_{\mathbb{R}^d}\mathrm{d}x\underbrace{G(x,y)}_{\mathbb{1}_A(x)-2\mathbb{1}_A(x)\mathbb{1}_A(x-y)+\mathbb{1}_A(x-y)}$$

$$=^\S 2\int_{\mathbb{R}^d}\mathrm{d}y j_\epsilon(y)\int_A\mathrm{d}x G(x,y)$$

$$=^\P 2\int_{\mathbb{R}^d}\mathrm{d}z j(x)\int_A\mathrm{d}x G(x,\epsilon z)$$

Thus it suffices to prove

$$\lim_{z\to 0}\int_A\mathrm{d}x\mathbb{1}_A(x-z)=\int_A\mathrm{d}x\quad(*)$$

because of $\left|\int_A\mathrm{d}x G(x,\epsilon z)\right|\le 2\int_A\mathrm{d}x,\ j\in L^1$ and dominated convergence.

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†Hölder with $p'=\frac{2}{p}(>1)$ and its conjugate q'

‡Substitute $y:=\frac{x}{\epsilon}$

3rd act: Since $\int_A \mathrm{d}x \mathbb{1}_A(x-z) \le \int_A \mathrm{d}x$, $(*)$ follows from $\liminf_{z\to 0} \underbrace{\int_A \mathrm{d}x \mathbb{1}_A(x-z)}_{=:I(z)} \ge \int_A \mathrm{d}x$ $(**)$ Lebesgue measure is outer regular (see Example 40) $\Rightarrow$ $\forall \delta > 0$, $\exists B \subseteq \mathbb{R}^d$ open such that $A \subseteq B$ and $\int_{B\setminus A} \mathrm{d}x \le \delta$

$$\Rightarrow I(z) = \int_A \mathrm{d}x \mathbb{1}_B(x-z) - \underbrace{\int_A \mathrm{d}x \mathbb{1}_{B\setminus A}(x-z)}_{\le \int_{\mathbb{R}^d} \mathrm{d}x \mathbb{1}_{B\setminus A}(x-z) \le \delta}$$

$$\ge \int_A \mathrm{d}x \mathbb{1}_B(x-z) - \delta$$

B open $\Rightarrow \lim_{z\to 0} \mathbb{1}_B(x-z) = \mathbb{1}_B(x) \quad \forall x \in B \Rightarrow$ † $\lim_{z\to 0} \int_A \mathrm{d}x \mathbb{1}_B(x-z) = \int_A \mathrm{d}x \Rightarrow \liminf_{z\to 0} I(z) \ge \int_A \mathrm{d}x - \delta \Rightarrow \forall \delta > 0 \Rightarrow (**)$

□

Proof of Thm 45. Let $\epsilon > 0$ and $f \in L^p(\Omega)$. For $n \in \mathbb{N}$ define $K_n := \{x \in \Omega\colon \operatorname{dist}(x, \delta\Omega) \ge \frac{1}{n}$ and $|x| \le n\}$ compact in $\Omega \Rightarrow \Omega = \bigcup_{n\in\mathbb{N}} K_n$ and $K_n \subseteq K_{n+1}$, monotonous convergence $\Rightarrow \lim_{n\to\infty} \|f\mathbb{1}_{K_n}\|_p = \|f\|_p \Rightarrow \exists n \in \mathbb{N}\colon \|f - f\mathbb{1}_{K_n}\|_p < \frac{\epsilon}{2}$ New: $\exists \delta > 0$ such that $j_\delta * (f\mathbb{1}_{K_n}) \in \mathcal{C}_C^\infty(\Omega)$ by Lemma 47 and $\|f\mathbb{1}_{K_n} - j_\delta * (f\mathbb{1}_{K_n})\|_p < \frac{\epsilon}{2}$ $\Rightarrow \|f - j_\delta * (f\mathbb{1}_{K_n})\|_p < \epsilon$ □

4. Decomposition of Measures

Notation. *$\mathcal{A}$ is a σ-Algebra on a space X, μ, ν are measures on $\mathcal{A}$*

3.52. Definition. i) μ is absolutely continuous with respect to $\nu (\mu \ll \nu) :\Leftrightarrow \nu(A) = 0$ for some $A \in \mathcal{A} \Rightarrow \mu(A) = 0$ "every ν-null set is a μ-null set"

ii) μ and ν are mutually singular, $\mu \perp \nu :\Leftrightarrow \exists A \in \mathcal{A} : \mu(A) = 0$ and $\nu(X \setminus A) = 0$ "μ is concentrated on $X \setminus A$, ν on A

iii) sum of measures $\mu + \nu\colon \mathcal{A} \to [0, \infty], A \mapsto \mu(A) + \nu(A)$ is a measure.

3.53. Theorem (Radon-Nikodym). *Let μ be a finite measure let ν be σ-finite. Then $\exists_1$ measure μ_{ac} and $\exists_1$ measure μ_s: $\mu = \mu_{ac} + \mu_s$, $\mu_{ac} \ll \nu$ and $\mu_s \perp \nu$ (hence also $\mu_{ac} \perp \mu_s$). Moreover $\exists_1 h \in L^1(X,\nu)$: $\mu_{ac}(A) = \int_A d_\nu h \quad h = "\frac{d\mu_{ac}}{d\nu}" \ge 0$ is called the Radon-Nikodym derivative or*

3.54. Corollary. *In the situation of Thm. 51: $\mu \ll \nu \Leftrightarrow \mu$ has a density with respect to ν.*

3.55. Remark. Thm 51 can be extended to σ-finite measures μ, but then we only get a locally integrable density $\frac{\mathrm{d}\mu_{\mathrm{ac}}}{\mathrm{d}\nu} \in L^1_{\mathrm{loc}}(X, \nu) = \{$equiv. classes$f : X \to \mathbb{K}$ measurable $: f|_A \in L^1(A, \nu) \forall A \in \mathcal{A} : \mu(A) < \infty\}$

†dominated convergence and $A \subseteq B$

CHAPTER 4

The Cornerstones of Functional Analysis

1. Hahn-Banach extension theorem

Needs Zorn's Lemma in its proof.

4.1. Definition. Let M be a set and $D \subseteq M \times M$. Write $x \leq y :\Leftrightarrow (x, y) \in D$ ($\leq$ binary relation)

i) $\leq$ is a partial ordering on M (M a partially ordered set) $:\Leftrightarrow$
   - $x \leq x \ \forall x \in M$ (reflexive)
   - if $x \leq y$ and $y \leq z \Rightarrow x = y$ (antisymmetry)
   - $x \leq y$ and $y \leq z \Rightarrow x \leq z$ (transitive)

ii) $x, y \in M$ comparable $:\Leftrightarrow x \leq y$ or $y \leq x$, incomparable $:\Leftrightarrow$ not comparable.

iii) $\leq$ is a total ordering $:\Leftrightarrow$ $\leq$ is a partial ordering and all $x, y \in M$ are comparable.

iv) Let $W \subseteq M$. $u \in M$ is an upper bound for $W :\Leftrightarrow x \leq u \ \forall x \in W$ (need not exist). $m \in W$ is a maximal element$of W$ $:\Leftrightarrow m \leq x \ \forall x \in W$ then $m = x$ (needs not exist. Not necessarily an upper bound).

4.2. Example.
- "$\leq$" is a total ordering on $\mathbb{R}$
- "$\subseteq$" is a partial ordering on $\mathcal{P}(X)$

4.3. Lemma (Zorn's Lemma (Axiom! Equivalent to the axiom of choice)). *Let $M \neq \varnothing$ be a partially ordered set. If every totally ordered subset $W \subseteq M$ has an upper bound, then M posseses a maximal element (not necessarily unique)*

4.4. Theorem. *Every Hilbert space $X \neq \{0\}$ has an ONBA (finishes proof of Thm. 2.49)*

Proof. Let $M \coloneqq \{\xi \subset X : \xi \text{ orthonormal}\}(\neq \varnothing!)$ with "$\subseteq$" as a partial ordering. Let $W \subset M$ be totally ordered $\Rightarrow \mathcal{W} \coloneqq \bigcup_{\xi \in W} \xi$ is an upper bound for W. $\overset{\text{Zorn}}{\Rightarrow} M$ has maximal Element $\mathcal{M}$. Need to show: $\mathcal{M}$ is an ONB of X! Suppose not: $\Rightarrow \exists 0 \neq x \in X : x \perp q \ \forall q \in \mathcal{M}$. $\Rightarrow \tilde{\xi} \coloneqq \left\{\frac{x}{\|x\|}\right\} \cup \mathcal{M} \in M$ and $\mathcal{M} \subsetneq \tilde{\xi}$. This is a contradiction, so $\mathcal{M}$ is maximal. □

4.5. Remark. Similar arguments prove Thm. 2.4. (Existence of Hamel basis) (c.f. exercise)

The next theorem ensures that dual spaces contain enough functionals with nice properties.

4.6. Theorem (Hahn Banach Thm (real version)). *Let X be an $\mathbb{R}$-vector space, $p: X \to \mathbb{R}$ convex ($p(\alpha x + (1-\alpha)y) \leq \alpha p(x) + (1-\alpha)p(y) \ \forall x, y \in X, \ \forall \alpha \in [0,1]$.) Let $Y \subseteq X$ be a subspace, let $\lambda: Y \to \mathbb{R}$ linear with $\lambda(x) \leq p(x) \ \forall x \in Y$. Then $\exists \Lambda: X \to \mathbb{R}$ linear: $\Lambda|_Y = \lambda$ and $\Lambda(x) \leq p(x) \ \forall x \in X$*

Proof. **1$^{\text{st}}$ act:** Let $z \in X \setminus Y$ ($\neq \varnothing$ without loss of generality), $\overline{Y} \coloneqq \operatorname{span}(Y, z)$. Aim: Construct extension $\tilde{\lambda}$ of λ by suitably choosing $\tilde{\lambda}(z) \Rightarrow$

$\tilde\lambda(\alpha z + y) = \alpha\tilde\lambda(y) + \lambda(y)$; $\alpha \in \mathbb{R}$, $\forall y \in Y$. To find $\tilde\lambda(z)$ let $\alpha, \beta > 0$, $y_i \in Y$:

$$\begin{aligned}\beta\lambda(y_1) + \alpha\lambda(y_2) = \lambda(\beta y_1 + \alpha y_2) &= (\alpha+\beta)\lambda\left(\frac{\beta}{\alpha+\beta}y_1 + \frac{\alpha}{\alpha+\beta}y_2\right)\\ &\le (\alpha+\beta)p\left(\frac{\beta}{\alpha+\beta}(y_1 - \alpha z) + \frac{\alpha}{\alpha+\beta}(y_2+\beta z)\right)\\ &\le \beta p(y_2 - \alpha z) + \alpha p(y_2 + \beta z)\end{aligned}$$

$$\frac{1}{\alpha}[-p(y_1 - \alpha z) + \lambda(y_1)] \le \frac{1}{\beta}[p(y_2 + \beta z) - \lambda(y_2)]$$

$$\Rightarrow \exists \alpha \in \mathbb{R} : \sup_{\substack{y\in Y\\ \beta>0}} \frac{1}{\beta}\left[-p(y - \beta z) + \lambda(y)\right] \le a \le \inf_{\substack{y\in Y\\ \beta>0}} \frac{1}{\beta}\left[p(y+\beta z) - \lambda(y)\right] \qquad (*)$$

Definiere $\tilde\lambda(z) := a$. (*) guarantees $\tilde\lambda(\alpha z + y) \le p(\alpha z + y) \quad \forall \alpha \in \mathbb{R}, \ \forall y \in Y$. Thus $\tilde\lambda$ is the desired extension to $\tilde y$.

2nd act: Use Zorn's Lemma to get an extension to X. Let

$$\xi := \{\text{extensions } e \text{ of } \lambda \text{ with } e \le p \text{ on } \operatorname{dom}(e)\} \ne \varnothing \text{ (by above)}$$

Partial ordering $\le$ on ξ: $e_1 \le e_2 :\Leftrightarrow e_1 = e_2\|\operatorname{dom}(e)$. Let $W \subseteq \xi$ be totally ordered. Write $W = \{e_\alpha\}_{\alpha\in I}$ and define $\tilde e$: $\bigcup_{\alpha\in I}\operatorname{dom}(e_\alpha) \to \mathbb{R}$, $x \mapsto e_\beta(x)$ if $x \in \operatorname{dom}(e_\beta)$ for some $\beta \in I \Rightarrow \tilde e \in \xi$ because its linear and $\tilde e \le p$ on its domain $\Rightarrow \tilde e$ is upper bound for $W \overset{\text{Zorn}}{\Rightarrow} \exists$ maximales Element Λ in ξ, defined on subspace $\widetilde X \subset X$. We need to show $\widetilde X = X$.
Suppose not: then extend Λ as in 1st act to a subspace with one more dimension. This is a Contradiction and thus Λ maximal. □

The complex version of Hahn-Banach will reduce to the above.

4.7. Theorem (Hahn-Banach (complex)). *Let X be a $\mathbb{C}$-vector space, p: $X \to \mathbb{R}$, ($p(\alpha x + \beta y) \le |\alpha|p(x) + |\beta|p(y)$ $\forall x, y \in X$, $\alpha, \beta \in \mathbb{C}$, $|\alpha| + |\beta| = 1$. Let $Y \subseteq X$ be a subspace, let λ: $Y \to \mathbb{C}$ linear with $|\lambda(x)| \le p(x) \quad \forall x \in Y$. Then $\exists \Lambda$: $X \to \mathbb{C}$ linear: $|\Lambda|_Y = \lambda$ and $|\Lambda| \le p$ on X.*

Proof. Define $l(x) := \operatorname{Re}\lambda(x)$, $\mathbb{R}$-linear: l: $Y \to \mathbb{R}$, p fulfills assumptions of Thm 6, applying of which grants us the existence of L: $X \to \mathbb{R}$, $\mathbb{R}$-linear $L|_Y = l$ and $L \le p$ on X. Note $\lambda(x) = l(x) - il(ix) \quad \forall x \in Y$, since by $\mathbb{C}$-linearity $\operatorname{Re}(i\lambda(x)) = -\operatorname{Im}(\lambda(x))$. Define $\Lambda(x) := L(x) - iL(ix)$, $\mathbb{R}$-linear: $X \to \mathbb{C}$: $\Lambda|_Y = \lambda$. Λ is $\mathbb{C}$-linear † and $|\Lambda| \le p$ on X (fix $x \in X$, let $\theta = \arg(\Lambda(x))$.Putting it all together, we get

$$|\Lambda(x)| = e^{-i\theta}\Lambda(x) = \Lambda(e^{-i\theta}x) \underset{\operatorname{Re}\Lambda = L}{\overset{\text{l.h.s.}\in\mathbb{R}}{=}} \Lambda(e^{-i\Theta}x) \overset{\text{above}}{\le} p(e^{-i\theta}x) \le p(x) \qquad \square$$

4.8. Corollary. *Let X be a normed space, $Y \subseteq X$ a subspace and $f \in Y^*$ then there is an extension $F \in X^*$, $F|_Y = f$ and $\|F\|_{X^*} = \|f\|_{Y^*}$*

Proof. Theorem 7 with $p(x) := \|f\|_{Y^*}\|x\| \ \forall x \in X$ (Rest exercise) □

4.9. Corollary. *Let X be a normed space, $x_0 \in X$ then there is an $f \in X^*$ with $f(x_0) = \|x_0\|$ and $\|f\|_{X^*} = 1$.*

Proof. Set $Y = \operatorname{span}\{x_0\}$ and φ: $Y \to \mathbb{C}$, $x = \alpha x_0 \mapsto \alpha\|x_0\|$ linear and use Corollary 8 to get the existence of an $f \in X^*$, $f|_Y = \varphi$, $\|f\|_{X^*} = \|\varphi\|_{Y^*} = 1$ □

4.10. Corollary. *Let X be a normed space, $Z \subsetneq X$ a subspace and $x_0 \in X \setminus Z$ with $\operatorname{dist}(x_0, Z) =: d$. Then exists a $f \in X^*$ with $f|_Z = 0$, $f(x_0) = d$ and $\|f\|_{X^*} \le 1$*

Proof. Exercise 38 (Choose $Y := \operatorname{span}(Z, x_0)$, define $\varphi : Y \to \mathbb{C}$ suitably and apply Cor. 8) □

† because $\Lambda(ix) = L(ix) - iL(-x) = i\Lambda(x)$

2. Three Consequences of Baire's Theorem

4.11. Theorem (Uniform Boundedness Principle, Banach Steinhaus). *Let X be a Banach space, Y a normed space and $\mathcal{F} \subseteq \mathrm{BL}_Y(X)$. If $\forall x \in X$ holds that $\sup_{T\in\mathcal{F}} \|Tx\| < \infty$ then $\sup_{T\in\mathcal{F}} \|T\|_{X\to Y} < \infty$.*

Proof. For $n \in \mathbb{N}$ define $A_n \coloneqq \{x \in X : \|Tx\| \leq n \ \forall T \in \mathcal{F}\}$. By assumption $X = \bigcup_{n\in\mathbb{N}} A_n$ and $A_n = \bar{A}_n \ \forall n \in \mathbb{N}$. So by the Bair theorem and Corollary 66 iiii) exists an $n_0 \in \mathbb{N}$ so that A_{n_0} has non empty interior†. So if $z \in B_r(0)$, $T \in \mathcal{F}$,

$$\|Tz\| \leq \|T(z + x_0)\| + \|Tx_0\| \leq n_0 + \|Tx_0\|$$

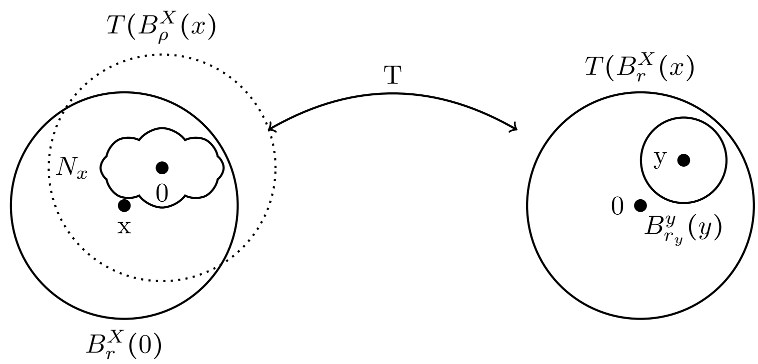
For all $x \in X \setminus \{0\}$ define $z \coloneqq \frac{r\cdot x}{2\|x\|} \in B_r(0)$, then the claim follows from

$$\|Tx\| \leq \frac{2\|x\|}{r}(n_0 + \|Tx_0\|) \quad \forall x \in X \quad \Rightarrow \quad \|T\| \leq \frac{2}{r}(n_0 + \|Tx_0\|) \quad \forall T \in \mathcal{F} \quad \square$$

4.12. Theorem (open mapping). *Let X, Y be Banach spaces and $T \in \mathrm{BL}_Y(X)$ which is onto. Then T is open.*

Proof. Let $A \subseteq X$ open. Need to show $T(A)$ open.

Claim 1: It suffices to show that for some $T > 0$, the image of the open Ball of radius r in X around 0, $T(B_r^X(0))$, has non-empty interior. For let $y = Tx \in T(B_r^X(0))$ be an interior point. Choose $r_y > 0 : B_{r_y}^Y(y) \subset T\left(B_r^X(0)\right)$. By continuouity of T $N_x \coloneqq T^{-1}\left(B_{r_y}^Y(y)\right) \subset B_r^X(0)$ is a neighbourhood of x, meaning there is a $\rho > 0 : N_X \subset B_\rho^X(x)$ and $T\left(B_\rho^X(x)\right) \supseteq B_{r_y}^Y(x)$



$$T\left(B_{r'}^X(x')\right) = T\underbrace{\left(B_{r'}^X(0)\right)}_{\frac{r'}{\rho}B_\rho^X(0)} + Tx' = \frac{r'}{\rho}\underbrace{\left(\underbrace{T\left(B_\rho^X(x)\right)}_{\supseteq B_{r_y}^Y(y)} - Tx\right)}_{\supseteq B_{r_y}^Y(0)} + Tx'$$

Thus: $\forall r' > 0 \ \exists r" > 0 \ \forall x' \in X \quad T\left(B_{r'}^X(x'')\right) \supseteq B_{r"}^Y(Tx') \qquad r'' \coloneqq \frac{r' r_y}{\rho}$.
This proves the claim.

Claim 2: $\boxed{\exists \epsilon > 0 : \ B_\epsilon^Y \subseteq \overline{T(B_1^X)}^{\ddagger}} \quad (*)$

Since T is onto $Y = \bigcup_{n\in\mathbb{N}} T(B_n^X)$ and using the completeness of Y and Baire's theorem $\overline{T(B_n^X)}$ has nonempty interior for some $n \in \mathbb{N}$. So for some

†Otherwise X meagre, i.e. $\exists x_0 \in A_{n_0}$ and $r > 0$: $B_r(x_0) \subset A_{n_0}$

‡Balls are centered at 0 unless indicated otherwise

$y \in \overline{T(B_n^X)}$ and $\epsilon > 0$: $B_\epsilon^Y(y) \subseteq \overline{T(B_n^x)}$ hence giving $B_\epsilon^Y(0) \subseteq \overline{T(B_n^x)} - y$, $y \coloneqq \lim_{n\to\infty} Tx_n$ for $(x_n) \subset B_n^X$ since

$$\overline{T(B_n^X)} - T(x_k) = \overline{T(\underbrace{B_n^X(-x_k)}_{B_{2n}})}$$

So $B_\epsilon^Y \subseteq \overline{T(B_{\epsilon_n}^X)}$, Rescaling the radius $(\frac{\epsilon}{2n})$ thus proves $(*)$.

Claim 3: $\overline{T(B_1^X)} \subset T(B_2^X)$ (so 2^{nd} and 1^{st} claim yield the Thm)

Indeed pick $y \in \overline{T(B_1^X)}$ then

$$\forall \epsilon > 0 : \exists x_1 \in B_1^X : y - Tx_1 \in B_{\frac{\epsilon}{2}}^Y \overset{(*)}{\subseteq} \overline{T(B_{\frac{1}{2}}^X)}.$$

Now pick $x_2 \in B_{\frac{1}{2}}^X$:

$$y - Tx_1 - Tx_2 \in B_{\frac{\epsilon}{2}}^Y \overset{(*)}{\subseteq} \overline{T(B_{\frac{\epsilon}{2}}^X)}.$$

Then apply induction induction to prove:

$$\forall n \in \mathbb{N} \exists x_n \in B_{2^{1-n}}^X : y - \sum_{j=1}^n Tx_j \in B_{2^{-n}\epsilon}^Y \qquad (**)$$

We then get $\sum_{n\in\mathbb{N}} \|x_n\| \le \sum_{n\in\mathbb{N}} 2^{-(n-2)} = 2$ so $\left(\sum_{n=1}^N x_n\right)_N$ is Cauchy in X. Since X is complete $\exists x \in X : x = \sum_{n\in\mathbb{N}} x_n \in B_2^X$.

$$(**) \Rightarrow 0 = \lim_{N\to\infty} \left\| y - T\left(\sum_{n=1}^N x_n\right)\right\| \overset{T \text{ and } \|\cdot\| \text{ cts}}{=} \|y - Tx\|$$

$\Rightarrow y = Tx \in T(B_2^X)$ □

4.13. Corollary (inverse mapping theorem). *Let X, Y be Banach Spaces and $T \in \mathrm{BL}_Y(X)$ a bijection, then $T^{-1} \in \mathrm{BL}_X(Y)$ (bounded!).*

Proof. T^{-1} exists, is linear and open by Theorem 12, so T^{-1} is continuous. □

Remark. The open and inverse mapping theorems deal with bounded operators. Applications deal with unbounded operators in many cases. "Closedness" makes it easier to handle them.

4.14. Definition. Let X, Y be normed spaces $T\colon \mathrm{dom}(T) \to Y$ linear with $\mathrm{dom}(T) \subseteq X$

i) The graph of T is given by

$$\mathcal{G}(T) \coloneqq \{(x, Tx) \in X \times Y : x \in \mathrm{dom}(T)\} \subseteq X \times Y.$$

Identify $X \times Y \simeq X \oplus Y = \left\{\begin{pmatrix} x \\ y \end{pmatrix} : x \in X, y \in Y\right\}$.

This is a normed space with norm $\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|_\oplus \coloneqq \|x\|_X + \|y\|_Y$ †

ii) T is closed $:\Leftrightarrow$ $\mathcal{G}(T)$ is closed in $X \times Y$

4.15. Remark. i) T closed $\Leftrightarrow$ $(x_n)_n \subset \mathrm{dom}(T)$ with $x_n \xrightarrow{n\to\infty} x \in X$ and $Tx_n \xrightarrow{n\to\infty} y \in Y \Rightarrow x \in \mathrm{dom}(T)$ and $y = Tx$

ii) compare with definition of T continuous, where convergence of Tx_n must also be shown (and is not supposed as above!)

Relation between closedness and boundedness is illuminated by

† see also Def. 2.51 for the case of inner product spaces

4.16. THEOREM (Closed Graph Theorem). *Let X,Y be Banach spaces and $T\colon \mathrm{dom}(T) \to Y$ a closed linear operator with $\mathrm{dom}(T) \subseteq X$. Then*

$$\mathrm{dom}(T) \textit{ closed in } X \Leftrightarrow T \textit{ bounded}$$

PROOF. "⇒" Let $(x_n) \subseteq \mathrm{dom}(T)$ with $x_n \xrightarrow{n\to\infty} x \in X$. Since $(x_n)_n$ is then Cauchy and T being bounded $(Tx_n)_n$ is also Cauchy. Since Y is complete there has to be a $y \in Y : Tx_n \xrightarrow{n\to\infty} y$. Using the fact that T is closed we get $x \in \mathrm{dom}(T)$ (does not require X to be complete)

"⇐" Note: $X \times Y$ is complete with the norm in Definition 14(i), if X,Y are complete. Thus $\mathcal{G}(T)$ complete in $X \times Y$ and $\mathrm{dom}(T)$ complete in X by assumption. The Projection $P_1 : \mathcal{G}(T) \to X, (x,Tx) \mapsto x$ is bounded, because

$$\|P_1(x,Tx)\|_X = \|x\|_X \le \underbrace{\|x\|_X + \|Tx\|_Y}_{\|(x,Tx)\|_{X\times Y}}, \text{ i.e. } \|P_1\| \le 1$$

and bijective with inverse $P_1^{-1} : \mathrm{dom}(T) \to \mathcal{G}(T), x \mapsto (x,Tx)$ By Corollary 14 (bounded inverse theorem) we know P_1^{-1} is bounded, i.e.

$$\exists b > 0 : \left\|P_1^{-1}x\right\|_{X\times Y} \le b\,\|x\|_X \;\; \forall x \in \mathrm{dom}(T), \text{ i.e. } \|Tx\|_Y \le (b-1)\,\|x\|_X . \qquad \square$$

4.17. EXAMPLE. $X,Y = \mathcal{C}(\mathbb{R})$ with $\|\cdot\|_\infty$. Then $\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{C}^1(\mathbb{R}) \to \mathcal{C}(\mathbb{R}), f \mapsto f$ is closed, because let $(f_n)_n \subset \mathcal{C}^1(\mathbb{R})$ with $\lim_{n\to\infty}\|f_n - g\|_\infty = 0$ for some $g \in \mathcal{C}(\mathbb{R})$ and $\lim_{n\to\infty}\|f_n' - k\|_\infty = 0$ for some $k \in \mathcal{C}(\mathbb{R})$ so

$$\int_0^x \mathrm{d}t\, h(t) = \lim_{n\to\infty}\int_0^x \mathrm{d}t f_n'(t) = \lim_{n\to\infty}[f_n(x) - f_n(0)] = g(x) - g(0)$$

Hence $g(x) = g(0) + \int_0^x \mathrm{d}t h(t) \Rightarrow g \in \mathcal{C}^1(\mathbb{R})$ with $g' = h$.†

4.18. COROLLARY (Hellinger-Toeplitz theorem). *$\mathcal{H}$ a Hilbert space, $T : \mathcal{H} \to \mathcal{H}$ linear, symmetric, i.e.*

$$\langle x, Ty\rangle = \langle Tx, y\rangle \quad \forall x,y \in \mathrm{dom}(T) \quad (= \mathcal{H})$$

Then T is bounded.

PROOF. We show $\mathcal{G}(T)$ is closed and the claim then follows from Theorem 16. Suppose $(x_n, Tx_n) \xrightarrow{n\to\infty} (x,y) \in \mathcal{H}\times\mathcal{H}$

$$\forall z \in \mathcal{H} \quad \langle z, y\rangle = \lim_{n\to\infty}\langle z, Tx_n\rangle = \lim_{n\to\infty}\langle Tz, x_n\rangle = \langle Tz, x\rangle = \langle z, Tx\rangle$$

So $y = Tx$ and $(x,y) \in \mathcal{G}(T)$. $\square$

4.19. REMARK. $T\colon \mathrm{dom}(T) \to \mathcal{H}$ linear, symmetric and unbounded, therefore by Corollary 18 $\mathrm{dom}(T) \subsetneq \mathcal{H}$ (ubiquitous in QM!)

3. (Bi-)Dual space and weak topologies

4.20. THEOREM. *Let X be a normed space. Then $\forall x \in X$*

$$\|x\| = \sup_{0\ne f\in X^*}\frac{|f(x)|}{\|f\|_{X^*}}$$

See Exercise 21 a) for the special case $X = l^P$

PROOF.
- $|f(x)| \le \|f\|_{X^*}\|x\|$ by definition of $\|f\|_{X^*}$

$$\Rightarrow S \coloneqq \sup_{0\ne f\in X^*}\frac{|f(x)|}{\|f\|_{X^*}} \le \|x\|$$

†If $\mathcal{C}(\mathbb{R})$ were replaced by $L^2(\mathbb{R})$ and $\|\cdot\|_\infty$ by $\|\cdot\|_2$, then $\frac{\mathrm{d}}{\mathrm{d}x}$ is not closed, but has a closed extension.

- Cor. 9 $\Rightarrow \exists \tilde{f} \in X^*$ with $\|\tilde{f}\|_{X^*} = 1$ and $\tilde{f}(x) = \|x\|$
$$\Rightarrow S \geq \frac{|\tilde{f}(x)|}{\|\tilde{f}\|_{X^*}} = \frac{\|x\|}{1}$$
□

4.21. Definition. For a normed space X the bidual $X^{**} := (X^*)^*$ is the dual space of X^*.

4.22. Theorem. *Let X be a normed space. The canonical embedding $J\colon X \to X^{**}, x \mapsto Jx$, where $Jx\colon X^* \to \mathbb{K}$, $f \mapsto f(x)$, is linear and isometric. If J is also surjective, we say that X is reflexive.*

4.23. Remark.
i) every Hilbert space is reflexive (Riesz!), so is every finite-dimensional space (use dual basis as in linear algebra)
ii) l^p and L^p are reflexive for $p \in]1, \infty[$
iii) X reflexive $\overset{\text{Cor. 2.34}}{\Rightarrow}$ X is complete

Proof of Thm. 22.
- well-defined: Let $x \in X$, then $Jx \in X^{**}$, because Jx is linear and
$$\|Jx\|_{X^{**}} = \sup_{0 \neq f \in X^*} \frac{|(Jx)(f)|}{\|f\|_{X^*}} = \sup_{0 \neq f \in X^*} \frac{|f(x)|}{\|f_{X^*}\|} \overset{\text{Thm. 20}}{=} \|x\|$$
- In addition, $X \to X^{**}$, $x \mapsto Jx$ is linear and isometric. □

4.24. Theorem. *X a normed space. Then X^* separable $\Rightarrow$ X separable.*

Proof. Let $A := \{f_n \in X^* : n \in \mathbb{N}\}$ be dense. $\forall n \in \mathbb{N}$ pick $x_n \in X$, $\|x_n\| = 1$, such that $|f_n(x_n)| \geq \frac{\|f_n\|_{X^*}}{2}$. Let $D := \operatorname{span}_{\mathbb{K}}\{x_n : n \in \mathbb{N}\} \subset X$. We will show D dense. Suppose, not: Then there exists a $z \in X$ with $\operatorname{dist}(z, D) =: d > 0$ and (by Corollary 10) $f \in X^*$ with $f|_D = 0$ and $f(z) = 0$. A is dense in X^* so there is a $(f_{n_k})_k \subset A$ such that $\|f_{n_k} - f\|_{X^*} \xrightarrow{n\to\infty} 0$. But $\|f_{n_k} - f\|_{X^*} \geq |(f_{n_k} - f)(x_{n_k}| = |f_{n_k}(x_{n_k})| \geq \frac{\|f_{n_k}\|_{X^*}}{2}$ $\forall k \in \mathbb{N} \Rightarrow f_{n_k} \xrightarrow{k\to\infty} 0$ in X^* $\Rightarrow f = 0$. Contradiction
In all $\operatorname{span}_{\mathbb{Q}(\text{resp. } \mathbb{Q}+i\mathbb{Q})}\{x_n : n \in \mathbb{N}\}$ is countable and dense in X. □

4.25. Remark. The reverse implication in Theorem 25 does not hold, as l^1 is separable, but $l^\infty = (l^1)^*$ is not!

4.26. Definition. Let X be a $\mathbb{K}$-vector space and $\{p_\alpha\}_{\alpha \in I}$, a familiy of seminorms on X, s.t. $\forall x \in X$, $x \neq 0$, $\exists \alpha \in I$ with $p_\alpha(x) \neq 0$. For $x \in X$ the collection $\{U_{\alpha,r}(x) := x + U_{\alpha,r} : \alpha \in I,\ r > 0\}$ with $U_{\alpha,r} := \{y \in X : p_\alpha(y) < r\}$ defines the neighbourhood subbase† of x of the locally convex topology generated by $\{p_\alpha\}_{\alpha \in I}$

4.27. Remark.
i) $U_{\alpha,r}(x) = \{y \in X : p_\alpha(y - x) < r\}$
ii) $(*)$ implies that the locally convex topology is Hausdorff.
iii) Neighbourhoods are convex sets:
$$y_1, y_2 \in U_{\alpha,r}(x) \ \Rightarrow\ \lambda y_1 + (1-\lambda) y_2 \in U_{\alpha,r}(x) \quad \forall \lambda \in [0,1]$$

4.28. Definition. Let X be a normed space. The weak topology on X is the locally convex topology on X generated by the seminorms $\{p_f := |f(\cdot)|\}_{f \in X^*}$‡

4.29. Lemma. *Let X be a normed space. Then*
a) *the weak topology is the coarsest topology on X such that all $f \in X^*$ are continuous. In particular: if $A \subseteq X$, then: A weakly open $\Rightarrow$ A strongly open (i.e. with respect to norm topology)*

†i.e. finite intersections fo $U_{\alpha,r}$'s form a neighbourhood base
‡$(*)$ in Definition 26 follows from Corollary 9 and note that $U_{f,r} := \{x \in X : |f(x)| < r\}$

b) *If* $\dim X < \infty$*, then weak and strong topology coincide.*
c) $(x_n)_{n\in\mathbb{N}} \subset X$ *converges weakly to* $x \in X \Leftrightarrow f(x_n) \xrightarrow{n\to\infty} f(x) \quad \forall f \in X^*$ *in symbols:* $x_n \xrightarrow{w} x$, $x_n \xrightarrow{n\to\infty} x$, $\text{W-lim}_{n\to\infty} x_n = x$

Proof. a) • $f \in X^*$ continuous in some topology $\mathcal{T}$ on $X \Rightarrow |f|: X \to [0,\infty[$ continuous with respect to $\mathcal{T}$, thereby $\{y \in X : |f(y)| < r\} \in \mathcal{T}$ giving us that $\mathcal{T}$ is finer than the weak topology
• show $f \in X^*$ is weakly continuous: Let $G \subset \mathbb{K}$ be open, take an open cover with balls $G = \bigcup_{\gamma\in G} B_{r_\gamma}(\gamma)$, where r_γ are suitably chosen radii.

$$\begin{aligned} f^{-1}(G) &= \bigcup_{\gamma\in G} f^{-1}\left(B_{r_\gamma}(\gamma)\right) \\ &= \bigcup_{\gamma\in G\cap\operatorname{ran}(f)} \{x \in X : |f(x) - \gamma| < r_\gamma\} \\ &\overset{\gamma=f(x_\gamma)}{=} \bigcup_{\gamma\in G\cap\operatorname{ran}(f)} \underbrace{\{x \in X : |f(x - x_\gamma)| < r_\gamma\}}_{x_r + U_{f,r_\gamma}(0)} \text{ open in weak topology.} \end{aligned}$$

b) Exercise, use dual basis!
c) By definition of weak convergence:

$\forall f \in X^*\ \forall r > 0,\ x_n \in U_{f,r}(x) \Leftrightarrow |f(x_n) - f(x)| < r$, for almost all $n \in \mathbb{N}$ □

4.30. Remark. i) The weak topology is Hausdorff, so weak limits are unique.
ii) The weak topology on X is not 1^{st} countable if X is infinite dimensional, hence <u>not metrizable</u> (because by Problem 18 every Hamel basis of X^* is uncountable; see D.L. Cohn, Measure Theory, footnote on p. 293)
iii) From a): Strong convergence implies weak convergence with the same limit (but the other direction does not hold in general).
iv) in a Hilbert space X: $x_n \xrightarrow{w} x \overset{\text{Riesz}}{\Leftrightarrow} \langle y, x_n\rangle \xrightarrow{n\to\infty} \langle y, x\rangle,\ \forall x \in X$
v) in l^1: $x_n \xrightarrow{w} x \Leftrightarrow x_n \xrightarrow{n\to\infty} x$
[I. Schur: J. Reine Angewandte Mathematik. 151, 79-111 (1921)]

Alternative proof of non-metrixability if $\mathbf{dim}\,\boldsymbol{X} = \mathbf{0}$.

4.31. Lemma. *Let* $\dim X = \infty$ *and* $A \subseteq X$ *weakly open, then* A *is unbounded with respect to* $\|\cdot\|$ *on* X*.*

Proof. Exercise. □

Assume the weak topology is generated by the metric d_W, then the Balls $A_n := \{x \in X : \mathrm{d}_W(x,0) < \frac{1}{n}\}$ are weakly open $\forall n \in \mathbb{N}$ By application of the Lemma $\forall n \in \mathbb{N}$, $\exists x_n \in A_n$ with $\|x_n\| \geq n$. Thus $x_n \xrightarrow{w} 0$ by construction but $\sup_{n\in\mathbb{N}} \|x_n\| = \infty$. ↯ by Lemma 31. □

Weakly convergent sequences are strongly bounded.

4.32. Lemma. *Let* X *be a normed space and* $x_n \xrightarrow{w} x$*. Then* $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$ *and* $\liminf_{n\to\infty} \|x_n\| \geq \|x\|$

Proof. • Let $x_n \xrightarrow{w} x \Rightarrow \forall f \in X^*$ (fixed) $f(x_n) \to f(x)$

$$\Rightarrow \sup_{n\in\mathbb{N}} \underbrace{|f(x_n)|}_{\underbrace{(Jx_n)}_{\in(X^*)^*} \underbrace{(f)}_{\in X^*\text{(complete)}}} < \infty$$

Apply the uniform boundedness principle with $X \equiv X^*$, $Y \equiv \mathbb{K}$ (Thm. 12),

$$\sup_{n\in\mathbb{N}} \underbrace{\|Jx_n\|_{X^{**}}}_{\|x_n\|} < \infty$$

- $\liminf$: Corollary 9: $\Rightarrow \forall y \in X \exists f_y \in X^*$ with $\|f_y\|_{X^*} = 1$ and $f_y(y) = \|y\|$

$$\overset{y=x}{\Rightarrow} \|x\| = f_x(x) = \lim_{n\to\infty} \underbrace{f_x(x_n)}_{\le \underbrace{\|f_x\|_{X^*}}_{1}\|x_n\|} \le \liminf_{n\to\infty} \|x_n\| \qquad \square$$

A useful characterization of weak convergence

4.33. Lemma. *Let X be a normed space. Then*

$$x_n \xrightarrow{w} x \Leftrightarrow \begin{cases} \bullet & \sup_{n\in\mathbb{N}} \|x_n\| < \infty \\ \bullet & \exists F \subset X^*, \operatorname{span} F \text{ dense in } X^* : f(x_n) \xrightarrow{n\to\infty} f(x),\ \forall f \in F \end{cases}$$

Proof. "$\Rightarrow$" Clear by Lemma 31 and Lemma 29c)

"$\Leftarrow$" By an $\epsilon/3$-argument: let $g \in X^*$ and $\epsilon > 0$: Let $K > 0$: $\|x\| \le K$, $\sup_{n\in\mathbb{N}} \|x_n\| \le K$
- $\operatorname{span} F$ dense in $X^* \Rightarrow \exists f \in \operatorname{span} F$ such that $\|f - g\|_{X^*} \le \frac{\epsilon}{3K}$
- $f \in \operatorname{span}(F) \Rightarrow \exists N \in \mathbb{N}$ such that $\forall n \ge N\ \|f(x_n) - f(x)\| < \frac{\epsilon}{3}$ (true $\forall f \in F$ by assumption $\Rightarrow$ true for finite linear combinations)

$$\Rightarrow |g(x_n) - g(x)| \le \underbrace{|g(x_n) - f(x_n)|}_{|(g-f)(x_n)|} + |f(x_n) - f(x)| + \underbrace{|f(x) - g(x)|}_{|(f-g)(x)|}$$

$$\le \underbrace{\|g - f\|_{X^*}}_{\le \frac{\epsilon}{3K}} \underbrace{(\|x_n\| + \|x\|}_{\le 2K} + \frac{\epsilon}{3} \le \epsilon \qquad \square$$

4.34. Definition. Let X be a normed space. The weak*-topology on X^* is defined as the topology on X^* generated by the seminorms $p_x : f \mapsto |f(x)|\}_{x\in X}$

Warning. Not every normed space is a dual space of some other normed space, the weak*-topology is only for duals.

4.35. Lemma. *Let X be a normed space, then the weak*-topology on X^* is*

a) *Hausdorff*
b) *the coarsest topology on X^* such that all $f \in X^*$ are pointwise continuous*
c) *coarser than the weak topology on X^* and the two (weak and weak*) coincide if and only if X is reflexive*
d) *such that $(f_n)_{n\in\mathbb{N}} \subset X^*$ converges to $f \in X^*$ in the weak*-topology (in symbols: $f_n \xrightarrow{w^*} f$) if and only if $f_n(x) \xrightarrow{n\to\infty} f(x), \forall x \in X$*

Proof. a) Remark 27 II)
b) Analogous to proof of Lemma 29a)
c) From $X \subseteq (X^*)^*$ with equality if and only if X is reflexive
d) Analogous to proof of Lemma 29c) $\square$

There are analogous results to Lemmas 31 and 32 with analogous proofs.

4.36. Lemma. *Let X be a Banach space, $f \in X^*$, $(f_n)_{n\in\mathbb{N}} \subset X^*$*

a) *If $f_n \xrightarrow{w^*} f$, then $\sup_{n\in\mathbb{N}} \|f_n\|_{X^*} < \infty$ and $\liminf_{n\to\infty} \|f_n\|_{X^*} \ge \|f\|_{X^*}$*
b) *$f_n \xrightarrow{w^*} f$ holds if and only if $\sup_{n\in\mathbb{N}} \|f_n\|_{X^*} < \infty$ and if there exists an $A \subset X$, with $\operatorname{span}(A)$ dense in X and $f_n(x) \xrightarrow{n\to\infty} f(x),\ \forall x \in A$*

Importance of weak*-topology relies on

4.37. THEOREM (Banach-Alaoglu). *Let X be a Banach space. Then the closed unit ball in X^**

$$\widetilde{B}_1^* \coloneqq \{f \in X^* : \|f\|_{X^*} \leq 1\}$$

is weak-compact.*

PROOF. Let $A \coloneqq \bigtimes_{x\in X}\{z \in \mathbb{K} : |z| \leq \|x\|\}$ $f \in A$ is a mapping $X \to \mathbb{K}, x \mapsto f(x)$ with $|f(x)| \leq \|x\|, \forall x \in X$ By Tychonoff's Theorem (1.44), A is compact in the product topology (the coarsest topology such that $\pi_x : A \to \mathbb{K}, f \mapsto f(x)$ is continuous $\forall x \in X$) Now

$$\widetilde{B}_1^* = \{f \in A : f \text{ is linear}\} = \bigcap_{\substack{x,y\in X\\ \alpha,\beta\in\mathbb{K}}} A_{x,y;\alpha,\beta}$$

where

$$A_{x,y;\alpha,\beta} \coloneqq \{f \in A : f(\alpha x + \beta y) - \alpha f(x) + \beta f(y) = 0\} = \left(\pi_{\alpha x+\beta y} - \alpha\pi_x - \beta\pi_y\right)^{-1}(\{0\})$$

is closed in the product topology. Thus $\widetilde{B}_1^*$ is closed in the product topology and since a is comapct, $\widetilde{B}_1^*$ is compact in the product topology, but the product topology on $\widetilde{B}_1^*$ in weak* is not 1st-countable (c.f. Remark 30ii)), but ... □

4.38. THEOREM. *Let X be a separable normed space and let $K \subseteq X^*$. If K is weak*-compact, then the relative weak*-topology on K is metrizable.*

PROOF. See e.g. lecture notes of C.E. Heil (Thm. 3.48) `http://www.math.gatech.edu/~heil/6338/summer08/section9f.pdf` □

So by Thm. 1.40 i):

4.39. COROLLARY (Helly's theorem, version 2 of Banach-Alaoglu). *Let X be a separable Banach space. Then $\widetilde{B}_1^*$ is weak*-sequentially compact.*

ALTERNATIVE PROOF (WITHOUT THM. 37). Let $(f_n)_{n\in\mathbb{N}} \subset B_1^*$ and $\{x_k\}_{k\in\mathbb{N}} \subset X$ be dense. $\forall k \in \mathbb{N}$ fixed $(f_n(x_k))_n \subset \mathbb{K}$ is bounded granting us a convergent subsequence $(f_{n_k}(x_k))_k$ which by exploiting Cantor's diagonal sequence trick † gives us a subsequence $(m_j)_j \subset \mathbb{N}$ such that $\left(f_{m_j}(x_k)\right)_j$ converges $\forall k \in \mathbb{N}$. Let $g(x) \coloneqq \lim_{j\to\infty} f_{m_j}(x)$. $\forall x \in \operatorname{span}(\{x_k\}_k) \eqqcolon \operatorname{dom}(g)$

- $|g(x)| \leq \liminf_{j\to\infty} \left\|f_{m_j}\right\|_{X^*} \|x\| \leq \|x\|$, $\forall x \in \operatorname{dom}(g)$, since $\left\|f_{m_j}\right\|_{X^*} \leq 1$
- g is linear on $\operatorname{dom}(g)$
- $\operatorname{dom}(g)$ dense in X

Theorem 2.30 gives us a bounded linear extension G of g to X with $\|G\| = \|g\| \leq 1$
Lastly $f_{m_j} \xrightarrow{w^*} G \in \widetilde{B}_1^*$ as $j \to \infty$ by Lemma 35b) □

4.40. THEOREM. *Let X be a Banach space, then X reflexive $\Leftrightarrow X^*$ reflexive*

PROOF. "⇒" Exercise
"⇐" Follows from: "a closed subspace of a reflexive space is reflexive". See Werner Thm. III.3.4 for details. □

4.41. THEOREM (version 3 of Banach-Alaoglu). *Let X be a Banach space, then X reflexive $\Leftrightarrow \widetilde{B}_1 \coloneqq \{x \in X : \|x\| \leq 1\}$ weakly compact.*

PROOF. "⇒" Since X reflexive, the weak*-topology in $(X^*)^*$ is the weak topology in X and Thm. 36.
"⇐" See e.g. Dunford, Schwartz: Linear operators, Vol. I, Interscience, 1966, Thm. V.4.7. □

†c.f. proof of Theorem 1.57

Another deep fact:

4.42. Theorem (<u>Eberlein-Šmulian</u>). *Let X be a Banach space and $A \subseteq X$. Then*

$$A \text{ is weakly compact} \Leftrightarrow A \text{ is weakly sequentially compact}$$

Proof. R.B. Holmes, Geometric Functional Analysis and it's Applications, Springer 1975 or R. Whitby, Mathematische Annalen 172, 116-118(1967) □

4.43. Example. Implications of Banach-Alaoglu theorems for compactness of unit ball $\widetilde{B}_1$ in

	weak	weak*	weak sequentially	weak* sequentially
l^1 $(\simeq c_0^*)$	No	✓	No	✓
l^p $(\simeq l^{q^*})$	✓			
l^∞ $(\simeq l^{1^*})$	No	✓	No	✓
L^1 $(\nexists)$	No	n.a.	No	n.a.
L^p $(\simeq L^{q^*})$	✓			
L^∞ $(\simeq L^{1^*})$	No	✓	No	✓

here $p, q \in], \infty[, \frac{1}{q} + \frac{1}{p} = 1$

CHAPTER 5

Bounded Operators

1. Topologies on Bounded Operators

5.1. Definition. Let X, Y be normed spaces

i) uniform (operator) topology on $\mathrm{BL}_Y(X)$:= norm topology of $\|\cdot\|_{X\to Y}$

ii) strong (operator) topology on $\mathrm{BL}_Y(X)$:= locally convex topology generated by family of seminorms $\{p_x : T \mapsto \|Tx\|\}_{x\in X}$

iii) weak (operator) topology on $\mathrm{BL}_Y(X)$:= locally convex topology generated by family of seminorms $\{p_{l_x} : T \mapsto |l(Tx)|\}_{\substack{x\in X\\ l\in Y^*}}$

5.2. Remark. i) recall (Thm. 2. 32): If Y is a Banach space then $\mathrm{BL}_Y(X)$ is a Banach space with respect to the uniform operator topology.

ii) Seminorms 1. ii) and 1. iii) obey (*) in Definition 4. 26, thus strong and weak operator topology are Hausdorff (⇒ limits are unique)

iii) Topologies in 1. ii) and 1. iii) are not first countable. If X is infinite dimensional then we need to distinguish compactness, sequential compactness, etc.

5.3. Lemma. i) *The* $\left\{\begin{array}{c} \textit{strong} \\ \textit{weak} \end{array}\right\}$ *operator topology is the coarsest topology on* $\mathrm{BL}_Y(X)$ *such that all the maps*

$$\left\{\begin{array}{c} M_x : BL_Y(X) \to Y, T \mapsto Tx,\ x \in X \\ M_{l,x} : BL_Y(X) \to \mathbb{K}, T \mapsto l(Tx),\ x \in X, l \in Y^* \end{array}\right\}$$

are continuous.

ii)

Weak operator top.	⊆ *weak topology on* $\mathrm{BL}_Y(X)$	← *less important*
	⊆ *strong operator topology*	⊆ *uniform operator top.*
← *coarser*		*finer* →

iii) *Let* $T,\ T_n \in \mathrm{BL}_Y(X)$, $\forall n \in \mathbb{N}$, *then* $(T_n)_n$ *converges to* T *in the* $\left\{\begin{array}{c} \textit{strong} \\ \textit{weak} \end{array}\right\}$ *operator topology, in brief*

$$\left\{\begin{array}{c} \text{“strongly”: } T_n \xrightarrow{s} T \\ \text{“weakly”: } T_n \xrightarrow{w} T \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} \|T_n x - Tx\| \xrightarrow{n\to\infty} 0,\ \forall x \in X\} \\ l(T_n x - Tx) \xrightarrow{n\to\infty} 0,\ \forall x \in X,\ \forall l \in Y^*\} \end{array}\right\}$$

Proof. i) Analogous to proof of Lemma 4.29a)

iii) Analogous to proof of Lemma 4.29c)

ii) Upper branch: $M_{l,x} \in (\mathrm{BL}_Y(X))^*$. Lower branch: $\forall x \in X$ M_X is linear and bounded:

$$\|Mx\|_{\mathrm{BL}_Y(X)\to Y} = \sup_{0\neq T\in \mathrm{BL}_Y(X)} \frac{\|MxT\|_Y}{\|T\|_{X\to Y}} = \sup_{0\neq T\in \mathrm{BL}_Y(X)} \frac{\|Tx\|_Y}{\|T\|_{X\to Y}} \leq \|x\|$$

⇒ M_x continuous, if $\mathrm{BL}_Y(X)$ is equipped with the uniform topology $\overset{i)}{\Rightarrow}$ the uniform topology is finer than the strong operator topology. The fact, that the strong operator topology is finer than the weak one, follows from M_x continuous $\Rightarrow M_{l,x}$ continuous $\forall l \in Y^*$ □

5.4. Remark. Operator multiplication is "separably continuous", i.e. $\forall B \in \mathrm{BL}_Z(Y)$ $\mathrm{BL}_Y(X) \to \mathrm{BL}_Z(X), A \mapsto BA$ is continuous, in the uniform, weak and strong operator topologies, so is $\mathrm{BL}_Z(Y) \to \mathrm{BL}_Z(X), B \mapsto BA,\ \forall A \in \mathrm{BL}_Y(X)$, but $\mathrm{BL}_Z(Y) \times \mathrm{BL}_Y(X) \to \mathrm{BL}_Y(X), (B, A) \mapsto BA$ is "jointly" continuous in the uniform operator topology and jointly sequentially continuous in the strong operator topology (See Reed, Simon, Section 6)

5.5. Lemma. *Let $\mathcal{H}$ be a Hilbert space and $(T_n)_{n\in\mathbb{N}} \subset \mathrm{BL}(\mathcal{H})$.*

a) *If $(T_n x)_n \subset \mathcal{H}$ Cauchy $\forall x \in \mathcal{H} \Rightarrow \exists\, T \in \mathrm{BL}(\mathcal{H})$: $T_n \xrightarrow{s} T$*
b) *If $(\langle y, T_n x\rangle)_n \subset \mathbb{K}$ Cauchy $\forall x, y \in \mathcal{H} \Rightarrow \exists\, T \in \mathrm{BL}(\mathcal{H})$: $T_n \xrightarrow{w} T$*

Proof. a) By assumption $\forall x \in \mathcal{H},\ \exists y_x \in \mathcal{H}\colon T_n x \xrightarrow{n\to\infty} y_x$ in $\mathcal{H}$ define $T\colon \mathcal{H} \to \mathcal{H}, x \mapsto y_x$. It is linear and $T_n x \xrightarrow{n\to\infty} Tx,\ \forall x \in \mathcal{H}$
For all $x \in \mathcal{H}$ we have $\sup_{n\in\mathbb{N}} \|T_n x\| < \infty$. Together with Thm 3.11 this implies uniform boundedness. $S \coloneqq \sup_{n\in\mathbb{N}} \|T_n\| < \infty$

$$\Rightarrow \|T\| = \sup_{\substack{x\in\mathcal{H}\\ \|x\|=1}} \|Tx\| = \sup_{\substack{x\in\mathcal{H}\\ \|x\|=1}} \lim_{n\to\infty} \|T_n x\| = \sup_{\substack{x\in\mathcal{H}\\ \|x\|=1}} \lim_{n\to\infty} \|T_n\|\,\|x\| \le S$$

$\Rightarrow T \in \mathrm{BL}(\mathcal{H})$

b) By assumption

$$\forall x, y \in \mathcal{H}\ \sup_n |\langle y, T_n x\rangle| = \sup_n |\langle T_n x, , \rangle\, y| = \sup_n |l_n(y)| < \infty^{\dagger}$$

By uniform boundedness $\sup_n \|l_n\|_{\mathcal{H}^*} = \sup_n \|T_n x\| < \infty$. Using unifrom boundedness again we get $S < \infty$ (as in a)). Define $Q\colon \mathcal{H}\times\mathcal{H} \to \mathbb{K}, (x, y) \mapsto \lim_{n\to\infty} \langle T_n x, y\rangle$ sesquilinear, with

$$|Q(x,y)| \le \liminf_{n\to\infty} \underbrace{\|T_n x\|}_{\le \|T_n\|\|x\|} \|y\| \le S\,\|x\|\,\|y\|$$

$\Rightarrow {}^{\ddagger}\exists_1 T \in \mathrm{BL}(\mathcal{H})$ with $Q(x,y) = \langle Tx, y\rangle$ (i.e. $T_n \xrightarrow{w} T$) and $\|T\| \le S$
[A generalisation of LM 5 to Banach Spaces holds (Reed/Simon, Sec. 6.)] □

5.6. Example. For $BL(l^2)$

(1) For $x = (x_1, x_2, \dots) \in l^2$ and $n \in \mathbb{N}$ let $T_n x \coloneqq (\frac{1}{n}x_1, \frac{1}{n}x_2, \dots)$. So $(T_n)_n \subset \mathrm{BL}(l^2)$ and $T_n \xrightarrow{n\to\infty} 0$ uniformly.
(2) Let $T_n x \coloneqq (0, \dots, 0, x_{n+1}, x_{n+2}, \dots)$. Then $T_n \xrightarrow{s} 0$, because

$$\|T_n x\|^2 = \sum_{j=n+1}^{\infty} |x_j|^2 \xrightarrow{n\to\infty} 0,\ \forall x \in l^2$$

but T_n does not converge uniformly, because

$$\|T_n e_{n+1}\| = 1 \Rightarrow \|T_n\| \ge 1,\ \forall n \in \mathbb{N}$$

(3) Let $T_n x \coloneqq (\underbrace{0, \dots, 0}_{\text{n places}}, x_1, x_2, \dots)$. Then $T_n \xrightarrow{w} 0$, because

$$\langle y, T_n x\rangle = \left|\sum_{j=1}^{\infty} \overline{y}_{n+j} x_j\right| \le \|x\| \left(\sum_{j=1}^{\infty} |y_{n+j}|^2\right)^{\frac{1}{2}} \xrightarrow{n\to\infty} 0,\ \forall x, y \in l^2$$

but T_n does not converge strongly, because $\|T_n x\| = \|x\|,\ \forall x \in l^2$

† (because of convergence)
‡ By Exercise 29, itself a consequence of Riesz!

2. Adjoint Operators

5.7. DEFINITION. a) Let X, Y be normed spaces and $T \in \mathrm{BL}_Y(X)$. Then $T^\dagger : Y^* \to X^*, l \mapsto T^\dagger l$ where $(T^\dagger l)(x) := l(Tx)$ is a linear operator, the (Banach) adjoint of T.

b) Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $T \in \mathrm{BL}_{\mathcal{H}_2}(\mathcal{H}_1)$, then $T^* : \mathcal{H}_2 \to \mathcal{H}_1, y \mapsto T^* y$ where $T^* y \in \mathcal{H}_1$ is uniquely determined (Riesz!) from

$$\underbrace{\langle y, Tx\rangle_{\mathcal{H}_2}}_{l(x)\in\mathcal{H}_1^*} = \langle T^* y, x\rangle_{\mathcal{H}_1} = \langle z, x\rangle_{\mathcal{H}_1} \quad \forall x \in X, \text{ with } z := Ty$$

is a linear operator, the (Hilbert) adjoint of T.

5.8. WARNING. Even if $X = \mathcal{H}_1, Y = \mathcal{H}_2$ in 7a), then $T^\dagger \neq T^*$, because in b) duals $\mathcal{H}_j^*$ are identified with $\mathcal{H}_j$. $T^* = \mathcal{A}_1 T^\dagger \mathcal{A}_2^{-1}$ with $\mathcal{A}_j : \mathcal{H}_j^* \to \mathcal{H}_j$ is an anti-linear, bijective isometry (see Corollary 2.56)

5.9. REMARK. T^* in Definition 7b) is a linear operator, because $\forall x \in X$

$$\underbrace{\langle \alpha y + \beta z, Tx\rangle_{\mathcal{H}_2}}_{\langle T^*(\alpha y+\beta z), x\rangle_{\mathcal{H}_1}} = \overline{\alpha}\underbrace{\langle y, Tx\rangle_{\mathcal{H}_2}}_{\langle T^* y, x\rangle_{\mathcal{H}_1}} + \overline{\beta}\underbrace{\langle z, Tx\rangle_{\mathcal{H}_2}}_{\langle T^* z, x\rangle_{\mathcal{H}_1}} = \langle \alpha T^* y + \beta T^* z, x\rangle_{\mathcal{H}_1}$$

(analogous for $T^\dagger$ in a))

5.10. EXAMPLE. i) Let $X = Y = l^1$ and $Tx := (0, x_1, x_2, \dots)$ the right shift operator. Then the adjoint is $T^\dagger : {l^1}^* \to {l^1}^*$, and if $f \in {l^1}^*$ then there exists $\xi \in l^\infty : f(x) = \sum_{j\in\mathbb{N}} \xi_j x_j,\ \forall x \in l^1$, so $(T^\dagger f)(x) = f(Tx) = \sum_{j\in\mathbb{N}} \xi_{j+1} x_j$ $\Rightarrow T^\dagger f$ corresponds to $(\xi_2, \xi_3, \dots) \in l^\infty$

ii) For $\phi, \psi \in \mathcal{H}$ let $P_x := \psi\langle \phi, x\rangle, \forall x \in \mathcal{H}$, i.e. $P = \psi\langle\phi,\cdot\rangle \in \mathrm{BL}(\mathcal{H}) \Rightarrow P^* = \phi\langle\psi,\cdot\rangle \in \mathrm{BL}(\mathcal{H})$ because

$$\langle y, Px\rangle = \langle y, \psi\langle\phi, x\rangle\rangle = \langle y,\psi\rangle\langle\phi,x\rangle = \langle\langle\psi,y\rangle, x\rangle = \langle P^* y, x\rangle$$

5.11. THEOREM. *Let X, Y be normed spaces and $T \in \mathrm{BL}_Y(X)$. Then*

$$\|T\|_{X\to Y} = \|T^\dagger\|_{Y^*\to X^*}$$

PROOF.

$$\|T\|_{X\to Y} = \sup_{\substack{x\in X:\\ \|x\|=1}} \underbrace{\|Tx\|_Y}_{\sup_{\substack{l\in Y^*:\\ \|l\|=1}} |\underbrace{l(Tx)}_{(T^\dagger l)(x)}|} \overset{\text{Thm 4.20}}{=} \sup_{\substack{l\in Y^*\\ \|l\|=1}} \underbrace{\sup_{\substack{x\in X:\\ \|x\|=1}} \left|(T^\dagger l)(x)\right|}_{\|T^\dagger l\|_{X^*}} = \|T^\dagger\|_{Y^*\to X^*}$$

The interchange of suprema is allowed because:

$$\begin{aligned} \sup_x \sup_y A_{xy} &\le \sup_x \sup_y \sup_{x'} A_{x'y} = \sup_y \sup_{x'} A_{x'y} \\ &\le \sup_y \sup_{x'} \sup_{y'} A_{x'y'} = \sup_{x'} \sup_{y'} A_{x'y'} \\ \Rightarrow \sup_x \sup_y A_{xy} &= \sup_y \sup_x A_{xy} \end{aligned}$$ □

5.12. COROLLARY. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces and $T \in \mathrm{BL}_{\mathcal{H}-2}(\mathcal{H}_1)$. Then*

$$\|T\|_{\mathcal{H}_1\to\mathcal{H}_2} = \|T^*\|_{\mathcal{H}_2\to\mathcal{H}_1}$$

PROOF. Thm 11 and $\mathcal{A}_j$ in Warning 8 isometric and bijetive. □

5.13. THEOREM. *Let $\mathcal{H}$ be a Hilbert space and $T, S \in \mathrm{BL}(\mathcal{H})$*

a) *The mapping $\mathrm{BL}(\mathcal{H}) \to \mathrm{BL}(\mathcal{H})$, $T \mapsto T^*$ is an anti-linear, isometric isomorphism*

b) $(TS)^* = S^*T^*$
c) $(T^*)^* = T$
d) *If T has a bounded inverse T^{-1}, then T^* has a bounded inverse and* $(T^*)^{-1} = (T^{-1})^*$
e) $\|T^*T\| = \|T\|^2$

Proof. a) By definition of T^* and Corollary 12
b) and c) Exercise
d) $T^{-1}T = \mathbb{1} = TT^{-1} \overset{b)}{\Rightarrow} T^*(T^{-1})^* = \mathbb{1}^* = \mathbb{1} = (T^{-1})^*T^*$
e)
- $\|T^*T\| \le \|T^*\|\,\|T\| = \|T\|^2$
- $\|T^*T\| =^{\dagger} \sup_{0\ne x,y\in\mathcal{H}} \dfrac{\langle y, T^*Tx\rangle}{\|y\|\,\|x\|} \ge \sup_{0\ne x\in\mathcal{H}} \dfrac{\langle x, T^*Tx\rangle}{\|x\|^2} = \sup_{0\ne x\in\mathcal{H}} \dfrac{\|Tx^2\|}{\|x\|^2} = \|T\|^2$

□

5.14. Definition. Let $\mathcal{H}$ be a Hilbert space and $T \in \mathrm{BL}(\mathcal{H})$
- T is unitary $:\Leftrightarrow$ T bijective and $T^{-1} = T^*$ (i.e. $T^*T = TT^* = \mathbb{1}$)
- T is self-adjoint $:\Leftrightarrow T = T^*$
- T is normal $:\Leftrightarrow TT^* = T^*T$

5.15. Remark. (1) self-adjoint or unitary $\Rightarrow$ normal
(2) T unitary $\Rightarrow \langle Tx, Ty\rangle = \langle x, y\rangle = \langle T^*x, T * y\rangle$, $\forall x, y \in \mathcal{H}$
(3) T self-adjoint $\Rightarrow \langle x, Ty\rangle = \langle Tx, y\rangle$ in particular: $\langle x, T * x\rangle \in \mathbb{R}$, $\forall x \in \mathcal{H}$
(Note: self-adjoint $\overset{T \text{ bounded}}{\Leftrightarrow}$ symmetric, see Corollary 4.18)
(4) T normal $\Rightarrow \|Tx\| = \|T^*x\|$ $\forall x \in X$, thus $\ker(T) = \ker(T^*)$

5.16. Example. i) $\mathcal{H} = L^2([0,1])$, $(Tf)(x) \coloneqq \int_0^1 \mathrm{d}y\, k(x,y) f(y)$ with $k \in \mathcal{C}\left([0,1]^2\right)$. If $k(x,y) = \overline{k(y,x)}$, $\forall x, y \in [0,1] \Rightarrow T$ self-adjoint.
ii) $T = z\mathbb{1}$, $z \in \mathbb{C} \Rightarrow T^* = \bar{z}\mathbb{1}$, self-adjoint $\Leftrightarrow z \in \mathbb{R}$

3. The Spectrum

Notation. *X is a Banach Space and $\mathcal{H}$ a Hilbert space over $\mathbb{C}$*

5.17. Definition. Let $T \in \mathrm{BL}(X)$.
- **resolvent set of T:** $\rho(T) \coloneqq \{z \in \mathbb{C} : T - z\mathbb{1}$ bijective$^{\ddagger}\}$
  resolvent of T: $R_z(T) \coloneqq (T - z)^{-1} \in \mathrm{BL}(X)$ for $z \in \rho(T)$
  spectrum of T: $\mathrm{spec}(T) \equiv \sigma(T) \coloneqq \mathbb{C} \setminus \rho(T)$
- If there is a $0 \ne x \in X : Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then x is an eigenvector of T with eigenvalue λ. The point spectrum of T is :
  $\mathrm{spec}_P(T) \equiv \sigma_p(T) \coloneqq \{$Eigenvalues of T$\} = \{z \in \mathbb{C} : T - z$ not injective$\}$
- continuous spectrum of T:
  $\mathrm{spec}_c(T) \equiv \sigma_c(T) \coloneqq \{z \in \mathbb{C} : T - z$ injective & $\mathrm{ran}(T - z) \ne X$, but dense$\}$
- residual spectrum of T:
  $\mathrm{spec}_r(T) \equiv \sigma_r(T) \coloneqq \{z \in \mathbb{C} : T - z$ injective, but $\mathrm{ran}(T - z)$ not dense$\}$

5.18. Lemma. *Let $T \in \mathrm{BL}(X)$. Then $\sigma(T) \equiv \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$. If X is finite dimensional, then $\sigma_c(T) = \sigma_r(T) = \varnothing$*[§]

Proof. Obvious. [$\sigma_r(T) = \varnothing$ for most T of interest.] □

5.19. Definition. Let $D \subset \mathbb{C}$ be a region and $x : D \to X, z \mapsto x(z)$.

---

† Theorem 4.20 and Riesz

‡ hence also $T - z \equiv T - z\mathbb{1}$ has a bounded Inverse by Cor. 4.13

§ $\sigma_r(T) = \varnothing$ for most T of interest

- x is <u>(strongly) analytic</u> and $z_0 \in D :\Leftrightarrow \lim_{h\to 0} \frac{x(z_0+h)-x(z_0)}{h}$ exists in X
- x is <u>weakly analytic</u> at $z_0 \in D :\Leftrightarrow \ \forall l \in X^*$ the mapping $D \to \mathbb{C}$, $z \mapsto l(x(z))$ is analytic at $z_0 \in D$
- x is strongly/weakly analytic in $D :\Leftrightarrow \forall z_0 \in D$: is strongly/weakly analytic at $z_0 \in D$

5.20. Remark. i) X-valued, strongly analytic functions have analogous properties to $\mathbb{C}$-valued functions, e.g. they have a power series expansion converging w.r.t. $\|\cdot\|$. (simply replace $|\cdot|$ by $\|\cdot\|$ in the theorems)
ii) A function is strongly analytic if and only if it is weakly analytic (for "$\Leftarrow$" see e.g. Reed/Simon Thm VI.4)

5.21. Theorem. *Let $T \in \mathrm{BL}(X)$*[†]*. Then $\rho(T)$ is open in $\mathbb{C}$ and $\rho(T) \to \mathrm{BL}(X)$, $z \mapsto R_z(T)$ is (strongly) analytic on each connected component of $\rho(T)$.*

For $\lambda, \mu \in \rho(T)$, $R_\lambda(T)R_\mu(T) = R_\mu(T)R_\lambda(T)$ *(they commute)*

and the <u>1st resolvent equation</u> holds:

$$R_\lambda(T) - R_\mu(T) = (\lambda-\mu)R_\lambda(T)R_\mu(T) \tag{1}$$

The Proof of Thm 18 uses

5.22. Lemma. *Let $T \in \mathrm{BL}(X)$, $\|T\| < 1$. Then $(\mathbb{1}-T)^{-1} \in \mathrm{BL}(X)$ and $(\mathbb{1}-T)^{-1} = \sum_{j=0}^\infty T^j$*[‡] *where $T^0 := \mathbb{1}$*

Proof. $S := \sum_{j=0}^\infty T^j \in \mathrm{BL}(X)$ exists, because $\left\|T^j\right\| \le \|T\|^j \ \forall j \in \mathbb{N}$ and $\|T\| < 1$. Now

$$(\mathbb{1}-T)\sum_{j=0}^N T^j = \mathbb{1} - T^{N+1} = \sum_{j=0}^N T^j(\mathbb{1}-T) \overset{N\to\infty}{\Rightarrow} (\mathbb{1}-T)S = \mathbb{1} = S(\mathbb{1}-T)$$

as $T^{N+1} \xrightarrow{N\to\infty} 0$ (from $\left\|T^{N+1}\right\| \le \|T\|^{N+1} \xrightarrow{N\to\infty} 0$) □

5.23. Corollary. *Let $T \in \mathrm{BL}(X)$. Then $\sigma(T) \subseteq \{z \in \mathbb{C} : |z| \le \|T\|\}$. In particular, $\rho(T) \ne \varnothing$.*

Proof. For $z \ne 0 : (T-z)^{-1} = \frac{1}{z}(\frac{T}{z} - \mathbb{1})^{-1} \in \mathrm{BL}(X)$ for $|z| > \|T\|$. □

Proof of 21.
- Eq(1):

$$\begin{aligned}(T-\lambda)(R_\lambda - R_\mu)(T-\mu) &= (\mathbb{1} - (T-\lambda)R_\mu)(T-\mu)\\ &= T-\mu-(T-\lambda)\\ &= \lambda-\mu\end{aligned}$$

Thus equation 1 follows by multiplication with R_λ (left) and R_μ (right).
- Interchange λ with μ in (1) to get: $R_\lambda R_\mu = R_\mu R_\lambda$
- Openness of $\rho(T)$: Let $\lambda \in \rho(T) \ne \varnothing$ and $z \in \mathbb{C}$, s. t. $|z-\lambda| < \frac{1}{\|R_\lambda\|}$

$$\Rightarrow T - z = T - \lambda - (z-\lambda) = (T-\lambda)\underbrace{(\mathbb{1} - (z-\lambda)R_\lambda)}_{=:v} \tag{2}$$

By Lemma 22: $V^{-1} \in \mathrm{BL}(X)$ and

$$V^{-1} = \sum_{j=0}^\infty (z-\lambda)^j R_\lambda^j \tag{3}$$

$\overset{2)}{\Rightarrow} V^{-1}R_\lambda = R_z \in \mathrm{BL}(X) \Rightarrow z \in \rho(T)$

[†] For a discussion of Banach-space valued functions of a complex variable, see e.g. Hille, Phillips: Functional Analysis and Semigroups (AMS, 1957) or Dunford, Schwarz: Linear Operators, Vol. 1, Sect III.14

[‡] This is a norm convergent series.

- $\boxed{R_z \overset{3)}{=} \sum_{j=0}^{\infty}(z-\lambda)^j R_\lambda^{j+1}}$ is a norm convergent power series, thus analytic. □

5.24. LEMMA. *Let $T \in \mathrm{BL}(X)$. Then $\sigma(T) \neq \varnothing$.*

PROOF. Let $z \in \mathbb{C}, |z| > \|T\|$. By Lemma 22

$$R_z(T) = \frac{-1}{z}\sum_{j=0}^{\infty}\left(\frac{T}{z}\right)^j \in \mathrm{BL}(X). \qquad \text{(\underline{Neumann Series})}$$

$$\Rightarrow \|R_z(T)\| \xrightarrow{|z|\to\infty} 0 \qquad (*)$$

If $\sigma(T) = \varnothing \Rightarrow \mathbb{C} \to \mathrm{BL}(X), z \mapsto R_z(T)$ is entire and bounded (by (*)). By Liouville's Theorem: $z \mapsto R_z(T)$ constant $\overset{(*)}{\Rightarrow} R_z(T) = 0,\ \forall z \in \mathbb{C}$. Contradiction! □

5.25. DEFINITION. Let $T \in \mathrm{BL}(X)$. Spectral radius of T: $r(T) := \sup_{\lambda\in\sigma(T)} |\lambda|$

5.26. THEOREM. a) *Let $T \in \mathrm{BL}(X)$, then $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|T^n\|^{\frac{1}{n}}$*

b) *Let $T \in \mathrm{BL}(\mathcal{H})$ normal, then $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \|T\| = r(T)$*

PROOF. a) **Equality of** lim **and** inf**:** Without loss of generality assume $T^n \neq 0\ \forall n \in \mathbb{N}$ (otherwise claim is clear). Let $a_n := \ln(\|T^n\|)$ then

$$a_{n_1+n_2} \leq a_{n_1} + a_{n_2}\ \forall n_1, n_2 \in \mathbb{N}$$

Fix an arbitrary $m \in \mathbb{N}$, write $n \in \mathbb{N}$, $n > m$, as $n = qm + r$, $q \in \mathbb{N}$, $r \in \{0, \dots, m-1\}$, then

$$\frac{a_n}{n} \leq \frac{qa_m + a_r}{qm + r} \leq \frac{a_m}{m} + \frac{a_r}{qm + r}$$

$\Rightarrow \limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_m}{m} \Rightarrow \limsup_{n\to\infty} \frac{a_n}{n} \leq \inf_{m\in\mathbb{N}} \frac{a_m}{m}$ but $\liminf_{n\to\infty} \frac{a_n}{n} \geq \inf_{m\in\mathbb{N}} \frac{a_m}{m} \Rightarrow$ limit exists and $\lim_{n\to\infty} e^{\frac{a_n}{n}} = \inf_{n\in\mathbb{N}} e^{\frac{a_n}{n}}$

$\boldsymbol{r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}}$ **:** Since $z \mapsto R_z(T)$ is analytic in $\mathbb{C} \setminus \{z \in \mathbb{C} : |z| \leq r(T)\}$ it follows[†] that$R_z(T)$ has a Laurent series expansion which converges absolutely for all $|z| > r(T)$. But this must be the Neumann series (Uniqueness of the Laurent series!)

$$R_z(T) = -\frac{1}{z}\sum_{j=0}^{\infty}\left(\frac{T}{z}\right)^j$$

Note that absolute convergence of the Laurent series fails on $\{|z| > r(T) - \epsilon\}$ $\forall \epsilon > 0$ by definition of $\sigma(T)$. In other words: $r(T)^{-1}$ is the radius of convergence of $\xi \mapsto \sum_{j=0}^{\infty} T^j \xi^j$. The claim then follows by Hadamard's root criterion:

$$\frac{1}{r(T)} = \limsup_{j\to\infty} \left\|T^j\right\|^{\frac{1}{j}}$$

b) $T \in \mathrm{BL}(\mathcal{H})$ normal, therefore

$$\left\|T^2\right\|^2 \overset{\text{Thm 13. e)}}{=} \left\|T^2\left(T^2\right)^*\right\| = \left\|(TT^*)(TT^*)^*\right\| \overset{13. \text{ e)}}{=} \|TT^*\|^2 \overset{\text{Thm 13. e)}}{=} \|T\|^2$$

so $\left\|T^2\right\| = \|T\|^2$ and then by induction: $\left\|T^{2^k}\right\| = \|T\|^{2^k}$, $\forall k \in \mathbb{N}$, The claim follows by choosing a subsequence $n_k := 2^k$ □

[†]Complex analysis, c.f. J. Conway, Functions of one complex variable, Thm V.1.11

4. Compact Operators

NOTATION. *X, Y are Banach spaces and $\mathcal{H}$ a Hilbert space over $\mathbb{C}$*

5.27. DEFINITION. Let $T \in \mathrm{BL}_Y(X)$. T is compact $:\Leftrightarrow$ $\forall A \subset X$ bounded and $\overline{T(A)}$ compact in Y (i.e. T(A) relatively compact).

5.28. REMARK. T compact $\Leftrightarrow$ $\forall (x_n)_{n\in\mathbb{N}} \subset X$ bounded $\Rightarrow (Tx_n)_n \subset Y$ has a convergent subsequence. (since compactnes $\Leftrightarrow$ sequential compactness in metric spaces).

5.29. EXAMPLE. i) $X = Y = (C([0,1]), \|\cdot\|_\infty)$ and

$$(Tf)(x) := \int_0^x dy k(x,y) f(y)$$

with $k \in C\left([0,1]^2\right)$ compact by Arzela-Ascoli (c.f. Exercise 15)

ii) Finite-range operators are compact Let $\dim(\mathrm{ran}(T)) < \infty$ i.e. $\exists J \in \mathbb{N}$ and $y_1, \ldots, y_J \in Y$ linearly independent such that $Tx_n = \sum_{j=1}^J \alpha_j(x_n) y_j,\ \forall n \in \mathbb{N}$ Cor. 4.10 $\Rightarrow \exists l_1, \ldots, l_J \in Y^* : l_K(y_j) = \delta_{kj}$

$$\Rightarrow |\alpha_j(x_n)| = |l_j(Tx_n)| \leq \|l_j\| \|T\| \|x_n\| \ \forall j \forall n$$

So: (x_n) bounded, making $(\alpha_j(x_n))_n \subset \mathbb{K}$ bounded $\forall j = 1, \ldots, J$ which grants us a common convergent subsequence.

5.30. THEOREM. *Let $T \in \mathrm{BL}_Y(X)$ be compact and $(x_n)_n \subset X$ such that $x_n \xrightarrow{w} x \in X$. Then $(Tx_n)_n \subset Y$ is strongly convergent to Tx.*

PROOF. Let $x_n \xrightarrow{w} x \in X \overset{\text{LM 4.32}}{\Rightarrow} \sup_n \|x_n\| < \infty$. Let $y_n := Tx_n$; $\forall n \in \mathbb{N}$, $y := Tx$, and $l \in Y^* \Rightarrow l(y_n) - l(y) = (T^* l)(x_n - x) \xrightarrow{n\to\infty} 0$ by assumption, i.e. $y_n \xrightarrow{w} y$. Assume $(y_n)_n$ does not converge strongly then there exists an $\epsilon > 0$ and a subsequence $(n_k)_k \subset \mathbb{N}$ such that $\|y_{n_k} - y\| > \epsilon$, $\forall k \in \mathbb{N}$ but $(x_{n_k})_k$ is bounded. Since T is compact, there exists a subsequence $y_{n_{k_l}} \xrightarrow{l\to\infty} \tilde{y} \neq y$. So by Remark.4.30iii) $y_{n_{k_l}} \xrightarrow{w} \tilde{y} \neq y$ as $l \to \infty$ ↯. □

5.31. THEOREM. *Let $T \in \mathrm{BL}_Y(X)$*

a) *Let $(T_n)_n \subset \mathrm{BL}_Y(X)$, T_n compact $\forall n \in \mathbb{N}$ and $\|T_n - T\| \xrightarrow{n\to\infty} 0$ (uniform convergence!), then T is compact.*
b) *T is compact $\Leftrightarrow T^*$ compact* (*Schauder's theorem*)
c) *Let Z be a Banach space and $S \in \mathrm{BL}_Z(Y)$, and if S or T is compact, then ST is compact.*

PROOF. a) Let $(x_m)_m \subset X$ be bounded. Without loss of generality assume $\|x_m\| \leq 1$ $\forall m \in \mathbb{N}$. T_n is compact therefore $(T_n x_m)_m$ has a convergent subsequence with limit $y_n \in Y$, $\forall n \in \mathbb{N}$. Using Cantor's diagonal trick we are granted a common convergent subsequence $(m_k)_k \subset \mathbb{N}$ such that $(T_n x_{m_k})_k \xrightarrow{k\to\infty} y_n$, $\forall n \in \mathbb{N}$. Let $\epsilon > 0$ and $N \in \mathbb{N}$ such that $\|T_n - T_{n'}\| < \epsilon$, $\forall n, n' \geq N$

$$\overset{n,n'\geq N}{\Rightarrow} \|y_n - y_{n'}\| \leq \|y_n - T_n x_{m_k}\| + \underbrace{\|T_n x_{m_k} - T_{n'} x_{m_k}\|}_{\leq \epsilon (\text{unif. in k})} + \|T_{n'} x_{m_k} - y_{n'}\|$$

Let $k \to \infty$ and we get $\|y_n - y_{n'}\| < \epsilon$, so $(y_n)_n$ is Cauchy $\Rightarrow y_n \xrightarrow{n\to\infty} y \in Y$
Claim: $Tx_{m_k} \xrightarrow{k\to\infty} y$, as $\forall \epsilon > 0$, $\exists n \in \mathbb{N}$ s. t. $\|T - T_n\| < \frac{\epsilon}{3}$ & $\|y_n - y\| < \frac{\epsilon}{3}$

$$\begin{aligned}\Rightarrow \|Tx_{m_k} - y\| &\leq \|Tx_{m_k} - T_n x_{m_k}\| + \|T_n x_{m_k} - y_n\| + \|y_n - y\| \\ &< \frac{2}{3}\epsilon + \underbrace{\|T_n x_{m_k} - y_n\|}_{<\frac{\epsilon}{3} \text{ for k sufficiently large}} < \epsilon\end{aligned}$$

$\Rightarrow T$ compact.

b) clear, because bounded operators preserve boundedness and convergence.
c) follows from Arzelá-Ascoli, see Werner, Theorem III.4.4. □

5.32. Theorem. *Let $\mathcal{H}$ be a separable Hilbert space. Then every compact $T \in \mathrm{BL}(\mathcal{H})$ is the uniform limit of a sequence of operators of finite rank.*

Proof. Let $\{\phi_j\}_{j\in\mathbb{N}} \subset \mathcal{H}$ be an ONB and

$$\lambda_n := \sup_{\substack{\psi \in \mathrm{span}(\phi_1,\ldots,\phi_n)^\perp \\ \|\psi\|=1}} \|T\psi\| \qquad \forall n \in \mathbb{N}$$

Thus $(\lambda_n)_n$ is non-negative and decreasing which means there is a limit $\lambda \geq 0$
Claim: $\lambda = 0$, because $\forall n \in \mathbb{N}$, $\exists \phi_n \in \mathrm{span}(\phi_1, \ldots, \phi_n)^\perp$, $\|\psi_n\| = 1$ s. t. $\|T\psi_n\| \geq \frac{\lambda}{2}$.
But $\psi_n \overset{w}{\Rightarrow} 0$ (since $|\langle y, \psi_n\rangle|^2 \leq \left(\sum_{j=n+1}^\infty |\langle \phi_j, y\rangle|^2\right) \underbrace{\|\psi_n\|^2}_{1} \xrightarrow{n\to\infty} 0, \forall y \in \mathcal{H}$) By Theorem 30 then $T\psi_n \xrightarrow{n\to\infty} 0 \Rightarrow \lambda = 0$. Let $R_n := \sum_{j=1}^n \langle \phi_j, \cdot\rangle T\phi_j \in \mathrm{BL}_Y(X)(\text{rank } \leq n)$

$$\begin{aligned}\Rightarrow (T - R_n)\psi &\overset{\text{Thm. 2.47}}{=} (T - R_n) = \sum_{l=1}^\infty \langle \phi_l, \psi\rangle \phi_l \\ &= \sum_{l=1}^\infty \langle \phi_l, \psi\rangle T\phi_l - \sum_{l=1}^n \langle \phi_l, \psi\rangle T\phi_l \\ &= \sum_{l=n+1}^\infty \langle \phi_l, \psi\rangle T\phi_l = T\Bigg(\underbrace{\sum_{l=n+1}^\infty \langle \phi_l, \psi\rangle \phi_l}_{\in \mathrm{span}(\phi_1,\ldots,\phi_n)^\perp, \text{with norm } \leq 1}\Bigg)\end{aligned}$$

$$\Rightarrow \|T - R_n\| = \sup_{\substack{\psi\in\mathcal{H} \\ \|\psi\|=1}} \|(T - R_n)\psi\| \leq \lambda_n \xrightarrow{n\to\infty} 0$$ □

5. Fredholm Alternative and Canonical Form for Compact Operators

Notation. *$\mathcal{H}$ is a separable Hilbert space over $\mathbb{C}$*

Motivation. *Let M be a (finite) quadratic matrix and $z \in \mathbb{C}$. Then either $M\psi = z\psi$ has a solution $\psi \neq 0$ or $(M - z)^{-1}$ exists (i.e. $M - z$ is bijective).*

Consequentially the inhomogenous equation $(M - z)\psi = \phi$ has a unique solution in the latter case; in the former case, there is no solution if $\phi \notin \mathrm{ran}(M - z)$, whereas for $\phi \in \mathrm{ran}(M - z)$ there exist infinitely many.
This is not true in general, e.g. for $\mathcal{H} = L^2([0, 1])$, $(M\psi)(x) := x\psi(x)$ and $z = 1$: $M\psi = \psi$ has no solution $0 \neq \psi \in \mathcal{H}$ but at the same time $M - \mathbb{1}$ is not surjective:

$$((M - \mathbb{1})\psi)(x) = (x - \mathbb{1})\psi(x) \Rightarrow x \mapsto |x - \mathbb{1}|^{-\frac{1}{3}} \notin \mathrm{ran}(M - \mathbb{1})$$

But for compact operators the Fredholm alternative holds! The following is the master result.

5.33. Theorem (Analytic Fredholm theorem). *Let $D \subseteq \mathbb{C}$ be open, connected. Let $f\colon D \to \mathrm{BL}(\mathcal{H})$ be analytic and $f(z)$ compact $\forall z \in D$. Then either*

A1) $(f(z) - \mathbb{1})^{-1}$ *does not exist for any* $z \in D$

or

A2) *There exists a discrete subset* $S \subset D$ *without accumulation points in* D *such that* $(f(z) - \mathbb{1})^{-1}$ *exists* $\forall z \in D \setminus S$. *Moreover* $z \mapsto (f(z) - \mathbb{1})^{-1}$ *is analytic in* $D \setminus S$ *and* $z \in S \Leftrightarrow f(z)\psi = \psi$ *has a solution* $0 \neq \psi \in \mathcal{H}$

PROOF. It suffices to prove that for all $z_0 \in D$ there exists a neighbourhood $\mathcal{N}(z_0)$ such that either A1) or A2) holds in $\mathcal{N}(z_0)$ (instead of 0) (Colour $\mathcal{N}(z_0)$ red or blue depending on whether A1) or A2) holds. If both colours show up in D then by connectedness there exists a $z' \in D$ and $\mathcal{N}(z')$ containing both colours ↯).

Fix $z_0 \in D$ arbitrary then there exists $r > 0$ such that $\|f(z) - f(z_0)\| < \frac{1}{2}$, $\forall z \in B_r(z_0)$. By Thm. 32 there exists $F \in \mathrm{BL}(\mathcal{H})$ with finite rank $N \in \mathbb{N}$: $\|f(z_0) - F\| < \frac{1}{2}$ thus $\|f(z) - F\| < 1 \overset{\text{Lm 22}}{\Rightarrow} D_r \ni z \mapsto (\mathbb{1} - f(z) + F)^{-1} \in \mathrm{BL}(\mathcal{H})$ is analytic (1)

There exists $\psi_1, \dots, \psi_N \in \mathcal{H}$ linearly independent and (see Example 29 ii) and Riesz) $\phi_1, \dots, \phi_N \in \mathcal{H}$ such that $F = \sum_{n=1}^N \langle \phi_n, \cdot \rangle \psi_n$. Let

$$\gamma_n(z) \coloneqq \left((\mathbb{1} - f(z) + F)^{-1}\right)^* \phi_n \in \mathcal{H}, \ \forall n = 1, \dots, N$$

$$\text{Thus } g(z) \coloneqq F(\mathbb{1} - f(z) + F)^{-1} = \sum_{n=1}^{N} \langle \gamma_n(z), \cdot \rangle \psi_n \tag{2}$$

$$\text{and hence } f(z) - \mathbb{1} = (g(z) - \mathbb{1})(\mathbb{1} - f(z) + F) \tag{3}$$

Therefore $z \in D_r$: $f(z) - \mathbb{1}$ is not invertible as thence is $g(z) - \mathbb{1}$ and as will be proven below: There exists a solution $0 \neq \psi \in \mathcal{H}$ to $f(z)\psi = \psi$ if and only if there exists a solution $0 \neq \phi \in \mathcal{H}$ to $g(z)\phi = \phi$

Now the problem has been reduced to the finite rank case:

$$\exists 0 \neq \phi \in \mathcal{H}\colon g(z)\phi = \phi \overset{2)}{\Leftrightarrow} \exists \beta_1, \dots, \beta_N \in \mathbb{C}(\text{not all zero}):\ \beta_n \coloneqq \sum_{n=1}^{N} \langle \gamma_n(z), \psi_m \rangle \beta_m$$
$$\Leftrightarrow \mathrm{d}(z) \coloneqq \det\left(A(z) - \mathbb{1}_{N \times N}\right) = 0$$
$$\text{where } A_{nm}(z) \coloneqq \langle \gamma_n(z), \psi_m \rangle \ n, m = 1, \dots, N$$

But $\mathrm{d}\colon D_r \to \mathbb{C}$ is analytic (because A_{nm} is analytic $\forall n, m$) So by the identity thm. it follows that <u>either</u> $S_r \coloneqq \{z \in D_r \colon \mathrm{d}(z) = 0\}$ has no acc. point in D_r <u>or</u> $S_r = D_r$

<u>Proof of (*)</u>:

- If $g(z)\phi = \phi \Leftrightarrow d(z) = 0$ for some $0 \neq \phi \in \mathcal{H}$ then $g(z) - \mathbb{1}$ not invertible.
- If $d(z) \neq 0$ then $(A(z) - \mathbb{1}_{N \times N})\beta = \xi$ has a solution $\beta \in \mathbb{C}^N$, $\forall \xi \in \mathbb{C}^N$. Choose $\xi_n \coloneqq \langle \gamma_n(x), \psi \rangle$ for $\psi \in \mathcal{H}$ fixed but arbitrary. Now $\tilde{\phi} \coloneqq -\psi + \sum_{n=1}^N \beta_n \psi_n$ solves $(g(z) - \mathbb{1})\tilde{\phi} = \psi$ because

$$(g(z) - \mathbb{1})\tilde{\phi} = \psi - g(z)\psi + (g(z) - \mathbb{1}) \sum_{m=1}^{N} \beta_m \psi_m$$
$$\overset{2)}{=} \psi + \sum_{n=1}^{N} \psi_n \left[\underbrace{\sum_{m=1}^{N} (A_{nm}(z) - \delta_{nm}) \beta_m - \langle \gamma_n(z), \psi \rangle}_{0} \right]$$

$\Rightarrow g(z) - \mathbb{1}$ is invertible. This proves (*).

Finally: 1), 3) and the analycity of g give us the domain of analyticity of $z \mapsto (f(z) - \mathbb{1})^{-1}$ as $D_r \setminus S_r$. □

5.34. COROLLARY (<u>Fredholm Alternative</u>). *Let* $T \in \mathrm{BL}(\mathcal{H})$ *be compact and* $z \in \mathbb{C} \setminus \{0\}$. *Then* <u>*either*</u> $(T - z)^{-1}$ *exists* <u>*or*</u> $T\psi = z\psi$ *has a solution* $0 \neq \psi \in \mathcal{H}$.

Proof. Let $D = \mathbb{C} \setminus \{0\}$ in Thm. 34 and $f(z) \coloneqq \frac{r}{2}$. [A1) does not hold for $|z| \geq \|T\|$ so A2) must hold] □

5.35. Corollary (Riesz-Schauder-Theorem). *Let $T \in \mathrm{BL}(\mathcal{M})$ be compact. Then $\sigma(T)$ has no accumulation point, except possibly $\lambda = 0$. Moreover any $0 \neq \lambda \in \sigma(T)$ is an eigenvalue of finite multiplicity (i.e. the corresponding Eigenspace is finite-dimensional).*

Proof. As above let $D \coloneqq \mathbb{C} \setminus \{0\}$ and $f(z) \coloneqq \frac{T}{z}$ in Theorem 34. Then $\mathbb{C} \setminus (\{0\} \cup S) \subseteq \rho(T)$ and $S \subset \sigma_P(T)$ and hence $\sigma(T) \setminus \{0\} = \sigma_P(t) \setminus \{0\}$ and 0 is the only possible accumulation point of $\sigma(T)$.
Finite multiplicity of $0 \neq \lambda \in \sigma_p(T)$: suppose not then there exists an infinite ONB $\{\psi_n\}_n$ such that $T\psi_n = \lambda\psi_n,\ \forall n \in \mathbb{N}$. $\sup_n \|\psi_n\| = 1$ and since T is compact $(\lambda\psi_n)_n$ has a convergent subsequence ↯. □

5.36. Remark. i) $\dim(\mathcal{H}) = \infty \Rightarrow 0 \in \sigma(T)$
ii) Cor. 35 and Cor. 36 have generalisations to Banach spaces and to $\mathbb{K} = \mathbb{R}$.

5.37. Corollary (Canonical form for compact generators). *Let $T \in \mathrm{BL}(\mathcal{H})$ be compact. Then there exist orthonormal sets $\{\psi_n\}_{n\in\mathbb{N}}$, $\{\phi_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ (not neccessarily ONB's!) and singular values $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ with $\lambda_n \xrightarrow{n\to\infty} 0$ such that*

$$T = \sum_{n\in\mathbb{N}} \lambda_n \langle \psi_n, \cdot \rangle \phi_n \qquad \text{(convergent in op. norm)}$$

Note: $\lambda_n > 0$ for only finitely many n possible!)

The proof relies on the following Corollary.

5.38. Corollary (Hilbert-Schmidt Theorem). *Let $T \in \mathrm{BL}(\mathcal{H})$ be compact and self-adjoint. Then there exists ONB $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ and a null sequence $(\lambda_n)_n \subset \mathbb{R}$ such that $T\psi_n = \lambda_n\psi_n\ \forall n \in \mathbb{N}$. Moreover, $T = \sum_{n\in\mathbb{N}} \langle \psi_n, \cdot \rangle \psi_n$.*

Let $x_\perp \in \mathcal{H}_0^\perp \Rightarrow x_0 \in \mathcal{H}_0$: $\langle x_0, Tx_\perp \rangle \overset{T=T^*}{=} \langle Tx_0, x_\perp \rangle = 0, \Rightarrow Tx_\perp \perp \mathcal{H}_0$

Proof. Choose ONB $\mathcal{B}_j$ in each finite dimensional (Cor. 36) eigenspace corresponding to eigenvalue $\mu_j \neq 0\ \forall j = 1, \ldots$ with $\mu_i \neq \mu_j$ Then, because $T = T^*$ eigenvectors corresponding to different eigenvalues are perpendicular, $\mathcal{B} \coloneqq \bigcup_{j=1} \mathcal{B}_j$ is an orthonormal set in $\mathcal{H}$. Let $\mathcal{H}_0 \coloneqq \overline{\mathrm{span}(B)} \Rightarrow T(\mathcal{H}_0) \subseteq \mathcal{H}_0$

$$\overset{T=T^*}{\Rightarrow} T(\mathcal{H}_0^\perp) \subseteq \mathcal{H}_0^\perp \qquad \Rightarrow \tilde{T} \coloneqq T|_{\mathcal{H}_0^\perp} \colon \mathcal{H}_0^\perp \to \mathcal{H}_0^\perp \text{ is self-adjoint and compact.}$$

But $\sigma(\tilde{T}) = \{0\} \Rightarrow$ spectral radius $r(\tilde{T}) = 0 \overset{\text{Thm } 26b)}{\Rightarrow} \|\tilde{T}\| = 0 \Rightarrow \tilde{T} = 0 \Rightarrow \mathcal{H}_0^\perp \subseteq \ker(T)$.
So if $B^\perp$ is any ONB of $\mathcal{H}_0^\perp$ then $B \cup B^\perp$ is an ONB of $\mathcal{H}$ consisting of eigenvectors

- $(\lambda_n)_n$ is a null sequence by Cor. 36
- $T = \sum_{n\in\mathbb{N}} \lambda_n \langle \psi_n, \cdot \rangle \psi_n$ holds because lhs and rhs agree on ONB $\{\psi_n\}$
- For uniform convergence of the sum over n see proof of Thm. 32.

□

Proof of Cor. 38. T is compact, thus by Thm. 31c) T^*T is compact and self-adjoint. Cor. 39 $\Rightarrow \exists$ orthonormal set $\{\psi_n\}_n \subset \mathcal{H}$ and $\{\tilde{\lambda}_n\} \subset]0, \infty[$ such that

$$T^*T = \sum_{n=1} \tilde{\lambda}_n \langle \psi_n, \cdot \rangle \psi_n$$

Note:

- Terms with $\tilde{\lambda}_n = 0$ need not be taken into account
- $\forall \psi \in \mathcal{H} \colon 0 \leq \|T\psi\|^2 = \langle \psi, T^*T \rangle \psi \sum_{n=1} \tilde{\lambda}_n |\langle \psi_n, \psi \rangle|^2$
- Thus

- $(\psi = \psi_n)\tilde{\lambda}_m \geq 0 \forall m$
- $\ker(T) = \overline{(\operatorname{span}\{\psi_n\}_n}^{\perp}$

Let $\lambda \coloneqq \tilde{\lambda}_n^{\frac{1}{2}}$ and $\phi_n \coloneqq \frac{1}{2n} T\psi_n,\ \forall n \Rightarrow T\psi_n = \lambda_n \phi_n$. Hence $T = \sum_{n=1} \lambda \langle \psi_n, \cdot \rangle \phi_n$ because for $\psi \in \mathcal{H}$ arbitrary write $\psi = \sum_{m=1} \underbrace{\gamma_m}_{\in \mathbb{C}} \psi_m + \underbrace{\psi_0}_{\in \ker(T)} \Rightarrow$ lhs and rhs have same action on ψ. □

If the sum over n has infinitely many terms, then the uniform convergence in $\mathrm{BL}(\mathcal{H})$ was shown in the proof of Theorem 32.